Convergent Generalized Monotone Splitting of Matrices

by

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ABSTRACT

Let B and T be \( n \times n \) real matrices and \( r \) an \( n \)-vector and consider the system \( u = BTu + r \). A new sufficient condition is given for the existence of a solution and convergence of a monotone process to a solution. The monotone process is a generalization of the Collatz-Schröder procedure.

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1. INTRODUCTION

Collatz–Schröder [1] consider the system

\[(1.1) \quad u = Tu + r\]

where \( T \) is a given \( n \times n \) real matrix and \( r \) a given \( n \)-vector and prescribe the monotone iterative process

\[(1.2) \quad \begin{pmatrix} v^{i+1} \\ w^{i+1} \end{pmatrix} = \begin{pmatrix} T_1 & -T_2 \\ -T_2 & T_1 \end{pmatrix} \begin{pmatrix} v^i \\ w^i \end{pmatrix} + \begin{pmatrix} r \\ r \end{pmatrix} \quad i = 0, 1, 2, \ldots\]

where \( T = T_1 - T_2, \; T_1 \geq 0 \) and \( T_2 \geq 0 \). A sufficient condition for the monotonicity and convergence of the above process is the existence of initial \( v^0, w^0 \) satisfying

\[(1.3) \quad v^0 \preceq v^1, \quad w^1 \preceq w^0, \quad v^0 \preceq w^0, \]

where \( v^1, w^1 \) are computed from (1.2). Condition (1.3) guarantees that (1.1) has a solution \( u \) such that

\[(1.4) \quad v^0 \preceq v^1 \preceq \cdots \preceq v^i \preceq \cdots \preceq u \preceq \cdots \preceq w^i \preceq \cdots \preceq w^1 \preceq w^0\]

and \( u = \lim_{i \to \infty} \frac{v^i + w^i}{2} \).

In this work we consider the system

\[(1.5) \quad u = Bu + r\]
where \( B \) is some \( n \times n \) real nonsingular matrix and prescribe the iterative process (2.3). Here however the splitting \( T = T_1 - T_2 \) is not monotonic with respect to the nonnegative orthant but with respect to the dual cone generated by the rows of \( B^{-1} \), that is:

\[
B^{-1}y \geq 0 \implies T_1y \geq 0 \quad \text{and} \quad T_2y \geq 0.
\]

In Collatz-Schröder [1], \( B^{-1} = I \). A sufficient condition for the monotonicity and convergence of the iteration (2.3) is the existence of \( v^0, w^0 \) satisfying (2.5). Condition (2.5) guarantees that (1.5) has a solution satisfying \( B^{-1}v \leq B^{-1}u \leq B^{-1}w \) and 

\[
u = \lim_{i \to \infty} \frac{v_i + w_i}{2}.
\]

By using Motzkin's theorem of the alternative for linear inequalities [4, 3] a sufficient condition for (2.5) can be obtained, condition (3.2). This condition insures the existence of a solution \( u \) to (1.5) and the convergence and monotonicity of the iterative process (2.3). When applied to \( Au = b \), condition (3.2) gives existence results such as (3.6): If \( A \geq -I \), and \( A'x \geq 0 \), \( x \geq 0 \) implies that \( bx = 0 \), then \( Au = b \) has a solution. (Here and throughout a prime denotes the transpose.)
2. THE MONOTONE SPLITTING AND ITS CONVERGENCE

We consider here the problem

\[(2.0) \quad u = BTu + r\]

where \(B\) and \(T\) are given \(n \times n\) real matrices with \(B\) nonsingular, and \(r\) is a given \(n\)-vector. We split the matrix \(T\) as follows

\[(2.1) \quad T = T_1 - T_2\]

and require that

\[(2.2a) \quad B^{-1}y \succeq 0 \quad \implies \quad T_1y \succeq 0 \quad \text{and} \quad T_2y \succeq 0,\]

or equivalently \([2]\) we require that

\[(2.2b) \quad T_1B \succeq 0 \quad \text{and} \quad T_2B \succeq 0\]

(To see the equivalence of \((2.2a)\) and \((2.2b)\) we note that if \((2.2b)\) holds then \(B^{-1}y \succeq 0\) implies \(T_1BB^{-1}y = T_1y \succeq 0\). Conversely if \((2.2a)\) holds then \(B^{-1}B = I \succeq 0\) implies \(T_1B \succeq 0\).)

We consider the iteration

\[(2.3) \quad \begin{bmatrix} v_{i+1} \\ w_{i+1} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} T_1 & -T_2 \\ -T_2 & T_1 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} + \begin{bmatrix} r \\ r \end{bmatrix} \]

and begin by establishing the following result.
(2.4) **Convergence and Existence Theorem** Let (2.1) and (2.2) hold. If there exist $v^0, w^0$ in $R^n$ such that

$$
\begin{bmatrix}
B^{-1} & 0 \\
0 & B^{-1}
\end{bmatrix}
\begin{bmatrix}
v^0 \\
w^0
\end{bmatrix}
= 0,
B^{-1}(w^0 - v^0) \geq 0,
$$

with $v^1, w^1$ computed from (2.3), then

$$
\begin{bmatrix}
B^{-1} & 0 \\
0 & B^{-1}
\end{bmatrix}
\begin{bmatrix}
v^i \\
w^i
\end{bmatrix}
= 0,
B^{-1}(w^i - v^i) \geq 0
$$

for $i = 0, 1, 2, \ldots$. In addition to system $u = BTu + r$ has a solution $u$ such that $B^{-1}v^i \leq B^{-1}u \leq B^{-1}w^i$ and $u = \lim_{i \to \infty} \frac{v^i + w^i}{2}$.

**Remark** If we set $B = I$, then we obtain the results of Collatz-Schröder [1].

**Proof** We first establish (2.6) by induction. Because of (2.5), (2.6) holds for $i = 0$. Assume now that (2.6) holds for $i$ and proceed to show that it also holds for $i + 1$. 

\[
\begin{bmatrix}
B^{-1} & 0 \\
0 & -B^{-1}
\end{bmatrix}
\begin{bmatrix}
v^{i+2} \\
-w^{i+1}
\end{bmatrix}
+ \begin{bmatrix}
v^i \\
-w^i
\end{bmatrix}
\begin{bmatrix}
B^{-1} & 0 \\
0 & -B^{-1}
\end{bmatrix}
\begin{bmatrix}
r \\
r
\end{bmatrix}
= \begin{bmatrix}
B^{-1} & 0 \\
0 & -B^{-1}
\end{bmatrix}
\begin{bmatrix}
v^{i+1} \\
-w^{i+1}
\end{bmatrix}
\]
(by 2.3)

\[
\begin{bmatrix}
T_1 & T_2 \\
T_2 & T_1
\end{bmatrix}
\begin{bmatrix}
v^i \\
-w^i
\end{bmatrix}
\begin{bmatrix}
B^{-1} & 0 \\
0 & -B^{-1}
\end{bmatrix}
\begin{bmatrix}
r \\
r
\end{bmatrix}
= \begin{bmatrix}
B^{-1} & 0 \\
0 & -B^{-1}
\end{bmatrix}
\begin{bmatrix}
v^{i+1} \\
-w^{i+1}
\end{bmatrix}
\]
(by 2.6 and 2.2a)

= 0
(by 2.3)

We also have
\[
B^{-1}(w^{i+1} - v^{i+1}) = (T_1 + T_2)(w^i - v^i)
\]
(by 2.3)

\[
\geq 0
\]
(by 2.6 and 2.2a)

Hence (2.6) holds for \(i + 1\) and the induction is complete. We now have from (2.6) that
\[
B^{-1} v \leq B^{-1} v \leq \cdots \leq B^{-1} v \leq \cdots \leq B^{-1} w \leq \cdots \leq B^{-1} w \leq B^{-1} w.
\]

Hence the monotone sequences \(\{B^{-1} v_k\}\) and \(\{B^{-1} w_k\}\) have limits \(a_k^*\) and \(b_k^*\), so that also the vector sequences \(\{B^{-1} v^i\}\) and \(\{B^{-1} w^i\}\)
have limits \( a^* \) and \( b^* \). From the continuity of the linear operator \( B \) we have also that the vector sequences \( \{v^i\} \) and \( \{w^i\} \) converge to \( v = Ba^* \) and \( w = Bb^* \). Hence from (2.3)

\[
B^{-1}v = T_1v - T_2w + B^{-1}r
\]

\[
B^{-1}w = -T_2v + T_1w + B^{-1}r
\]

By letting \( u = \frac{v+w}{2} \) we have that

\[
B^{-1}u = (T_1 - T_2)u + B^{-1}r = Tu + B^{-1}r.
\]

That \( B^{-1}v \leq B^{-1}u \leq B^{-1}w \) follows from

\[
B^{-1}u = B^{-1} \left( \frac{v+w}{2} \right) = \frac{a^* + b^*}{2}.
\]

Q.E.D.

3. SUFFICIENT CONDITIONS FOR SOLVING \( u = BTu + r \)

By using Motzkin's theorem of the alternative \([4,3]\) we give now a sufficient condition for the existence of \( v^0, w^0 \) satisfying (2.5) and hence for the existence of a solution of \( u = BTu + r \) and the convergence of the iterative process (2.3).

(3.1) Convergence and Existence Theorem Theorem (2.4) holds with assumption (2.5) replaced by

\[
\begin{align*}
\langle (-I + T_1B)'x + (T_2B)'y \rangle & \geq 0 \\
\langle (T_2B)'x + (-I + T_1B)'y \rangle & \geq 0
\end{align*}
\]

\[
\Rightarrow x B^{-1} r = 0
\]

\[
x, y \geq 0
\]
Proof: We have to show that (3.2) implies (2.5). Now (3.2) implies that

\[
\begin{align*}
\langle \begin{align*}
(-I + T_1B)'x + (T_2B)'y & \geq 0 \\
(T_2B)'x + (-I + T_1B)'y & \geq 0 \\
-xB^{-1}r + yB^{-1}r & > 0
\end{align*} \rangle \\
\end{align*}
\]

has no solution \((x, y)\)

which implies that

\[
\begin{align*}
\langle \begin{align*}
(-B^{-1} + T_1)'x + T_2'y - (B^{-1})'z & = 0 \\
-T_2x - (-B^{-1} + T_1)'y + (B^{-1})'z & = 0 \\
(B^{-1})'x - (B^{-1}r)'y + \eta & = 0
\end{align*} \rangle \\
\end{align*}
\]

has no solution \((x, y, z, \eta)\)

which by Motzkin's theorem is equivalent to

\[
\begin{align*}
\langle \begin{align*}
(-B^{-1} + T_1)v^0 - T_2w^0 + B^{-1}r\zeta & \geq 0 \\
T_2v^0 - (-B^{-1} + T_1)w^0 - B^{-1}r\zeta & \geq 0 \\
-B^{-1}v^0 + B^{-1}w^0 & \geq 0
\end{align*} \rangle \\
\end{align*}
\]

has a solution \((v^0, w^0, \zeta)\)

which is equivalent to
\[
\begin{pmatrix}
T_1 v^0 - T_2 w^0 + B^{-1} r - B^{-1} v^0 & \geq 0 \\
B^{-1} w^0 + T_2 v^0 - T_1 w^0 - B^{-1} r & \geq 0 \\
B^{-1} (w^0 - v^0) & \geq 0
\end{pmatrix}
\]
has a solution \((v^0, w^0)\)

which is equivalent to (2.5) having a solution. Q.E.D.

By observing that if the system \(u = BU + r\) has a solution \(\tilde{u}\)
then the system \(-u = -BU + r\) has a solution \(-\tilde{u}\) we obtain the
following result from Theorem (3.1) by appropriate modifications.

(3.3) **Convergence and Existence Theorem** Let (2.1), (2.2) and (3.2) hold. Then there exist \(v^0, w^0\) such that

\[
\begin{pmatrix}
B^{-1} & 0 \\
0 & B^{-1}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
1 \\
v^0 - v^0
\end{pmatrix} \\
\begin{pmatrix}
0 \\
w^0 - w^1
\end{pmatrix}
\end{pmatrix}
\leq 0, \quad B^{-1} (w^0 - v^0) \leq 0
\]

where \(v^1, w^1\) are computed from the iteration

\[
\begin{pmatrix}
v^{i+1} \\
w^{i+1}
\end{pmatrix}
= \begin{pmatrix}
B & 0 \\
0 & B
\end{pmatrix}
\begin{pmatrix}
T_1 & -T_2 \\
-T_2 & T_1
\end{pmatrix}
\begin{pmatrix}
v^i \\
w^i
\end{pmatrix}
- \begin{pmatrix}
r \\
r
\end{pmatrix}, \quad i = 0, 1, 2, \ldots
\]

which produces \(v^i, w^i\) satisfying

\[
\begin{pmatrix}
B^{-1} & 0 \\
0 & B^{-1}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
1 \\
v^i - v^i
\end{pmatrix} \\
\begin{pmatrix}
0 \\
w^i - w^{i+1}
\end{pmatrix}
\end{pmatrix}
\leq 0, \quad B^{-1} (w^i - v^i) \leq 0.
\]
In addition the system $-u = -BTu + r$ has a solution $u$ such that $B^{-1}v_i \geq B^{-1}u \geq B^{-1}w_i$ and $u = \lim_{i \to \infty} \frac{v_i + w_i}{2}$.

The following convergence and existence result for $Au = b$ is obtained from Theorem (3.3) above by setting $T_2 = 0$, $A = -B^{-1} + T_1$ and $b = B^{-1}r$.

(3.4) Convergence and Existence for $Au = b$ Consider the system $Au = b$ where $A$ is a given $n \times n$ matrix and $b$ is a given vector. Assume that $AB + I \geq 0$ for some nonsingular $n \times n$ matrix $B$. If

$$\begin{cases} (AB)x \geq 0 \\ x \approx 0 \end{cases}$$

then $Au = b$ has a solution $u$. This solution can be obtained from the iteration

$$\begin{bmatrix} v^{i+1} \\ w^{i+1} \end{bmatrix} = \begin{bmatrix} B(T_1v^i - b) \\ B(T_1w^i - b) \end{bmatrix}, \quad i = 0, 1, 2, \ldots$$

starting with $v^0, w^0$ which exist and satisfy

$$B^{-1}(v^1 - v^0) \leq 0, \quad B^{-1}(w^0 - w^1) \leq 0, \quad B^{-1}(w^0 - v^0) \leq 0.$$

This iterative process produces $v^i, w^i$ satisfying

$$B^{-1}(v^{i+1} - v^i) \leq 0, \quad B^{-1}(w^i - w^{i+1}) \leq 0, \quad B^{-1}(w^i - v^i) \leq 0.$$
\[ B^{-1}v^i \geq B^{-1}u \geq B^{-1}w^i \]

and \( u = \lim_{i \to \infty} \frac{v^i + w^i}{2} \).

(3.5) **Remark** If \( A \) is nonsingular, then by taking \( B = -A^{-1} \) we conclude from the above that \( Au = b \) has a solution for any \( b \).

(3.6) **Corollary** If we take \( B = I \) in (3.4) we have that for \( A \geq -I \), if \( A'x \geq 0, \ x \geq 0 \) implies \( bx = 0 \), then \( Au = b \) has a solution, and the iteration and monotonicity relations of (3.4) simplify accordingly. Similarly if we take \( B = -I \) we have that for \( A \leq I \), if \( A'x \leq 0, \ x \geq 0 \) implies \( bx = 0 \) then \( Au = b \) has a solution.
BIBLIOGRAPHY


