Computer Sciences Department
University of Wisconsin
1210 West Dayton Street
Madison, Wisconsin 53706

APPLICATIONS OF THE MAXIMUM PRINCIPLE TO SINGULAR PERTURBATION PROBLEMS*,4

Fred
by
F. W. Dorr(1), S. V. Parter(2)
and L. F. Shampine(3)

Technical Report #101

December 1970

FOOTNOTES

(1) Los Alamos Scientific Laboratory; Los Alamos, New Mexico.

(2) University of Wisconsin; Madison, Wisconsin.

(3) University of New Mexico; Albuquerque, New Mexico.

* This research was supported by the Atomic Energy Commission and the Office of Naval Research (Contract No. N00014-67-A-0128-0004.)

(4) This report is an expansion of the second author's invited symposium lecture at the October 1970 SIAM (Boston) meeting.

1. Introduction

The maximum principle is an important and powerful tool in the study of second order elliptic partial differential equations, and, several authors [11], [15], [30] have used this fundamental tool in the study of singular perturbation problems for linear partial differential equations. The work of Eckhaus and DeJager [11] is an exceptional example of the power of the maximum principle in such problems.

Since this basic and useful tool is not particularly emphasized in the study of singular perturbation problems, our goal is to exhibit its applicability and versatility by way of numerous examples. We only consider two-point boundary value problems for second order ordinary differential equations, although we examine nonlinearities, turning points, and, in the last section, coupled pairs of such equations.

Some of our results are new (see section 5 and theorem 4.1 in particular). In other instances we have studied interesting examples to demonstrate the many ways our estimates may be used. The reader interested in the general field of singular perturbations is urged to consult the books by Cole [7] and Wasow [28] and the excellent survey articles by O'Malley [19] and Vasil'eva [27]. The articles by Carrier [3] and Wasow [29] describe many interesting examples.

In Section 2 we state the basic maximum principle in the form in which it is used in the remainder of the paper. We then use it to derive some basic comparison theorems, which provide bounds on the solutions of linear and quasilinear problems. The next two sections give applications of these preliminary results to determining the asymptotic behavior of solutions to boundary value problems. Section 3 treats linear problems under various hypotheses, including turning point problems. Section 4 is concerned with quasilinear problems of the form

$$\begin{cases} \epsilon y''(t) + \alpha(t,y(t),\epsilon)\,y'(t) = \gamma(t,y(t),\epsilon) \quad (a < t < b) \\ y(a) = A(\epsilon), \quad y(b) = B(\epsilon)\,. \end{cases} \tag{1}$$

The principal result complements work of R. v. Mises [16] and Coddington and Levinson [5]. A complete analysis of the quasilinear problem

$$\epsilon y'' + yy' - y = 0$$

is made with the use of the maximum principle. Such problems may involve turning points.

The remaining three sections are independent of one another. They represent different directions which can be pursued from the basic material of the preceding sections.

The quasilinear problems of Section 4 do not permit y' to appear nonlinearly. In Section 5 we study nonlinear problems which are quadratic in y'. These nonlinear results are related to the results of Haber and Levinson [14]. As we have indicated, our methods of analysis could be used to discuss other nonlinearities. Indeed, the special problem (see [23])

$$(2) \qquad \begin{aligned} \epsilon y'' &= y' - (y')^3, \qquad 0 \leq t \leq 1 \\ y(0) &= 0, \quad y(1) = 1/2 \end{aligned}$$

is easily treated by these methods. However, we restrict ourselves to these quadratic nonlinearities because maximum principle estimates also enable one to prove the existence of solutions for $\epsilon > 0$. These results, which are rather general and technical, stand alone and are not used in the following sections.

The maximum principle is a (very important) special case of the general theory of differential inequalities. Section 6 demonstrates the application of more general comparison results to a variety of linear, nonlinear, and turning point problems. An interesting physical problem describing the combination of diffusion and flow in a tubular reactor is examined. Asymptotically convergent upper and lower bounds are used for numerical purposes, and compared with direct numerical integration solutions in the chemical engineering literature.

Finally, in section 7 we turn to systems of the form

$$(3) \qquad \begin{cases} u'' = f(t,u,v) \\ \epsilon v'' + g(t,u,u')\,v' - c(t,u,u')\,v = 0 \end{cases}$$

which were studied by Dorr and Parter [9]. The present discussion extends their results for some interesting examples.

2. Basic Comparison Results

We begin this section by recalling some basic facts about the maximum principle. These results are well known, and we refer the reader to Protter and Weinberger [25, Sec. 1.1] for proofs.

For a fixed $\epsilon > 0$, let L denote the differential operator

$$Ly(t) = \epsilon\, y''(t) + \alpha(t)y'(t) - \beta(t)y(t) \qquad (0 < t < 1).$$

We assume that $\alpha(t)$ and $\beta(t)$ are bounded on every interval $[a,b] \subset (0,1)$, and that $\beta(t) \geq 0$ for $0 < t < 1$. Suppose that $y(t) \in C^2(0,1)$ satisfies

$$(2.1) \qquad Ly(t) \geq 0 \qquad (0 < t < 1).$$

The maximum principle states that if $y(t)$ assumes a nonnegative maximum value M at a point $t \in (0,1)$, then $y(x) \equiv M$. In addition, suppose that $y(t) \in C^1[0,1]$ and $y(t)$ is not identically a constant. If $y(t)$ has a nonnegative maximum at $t = 0$, and if the function $\alpha(t) - t\beta(t)$ is bounded from below at $t = 0$, then $y'(0) < 0$. If $y(t)$ has a nonnegative maximum at $t = 1$, and if the function $\alpha(t) + (1 - t)\beta(t)$ is bounded from above at $t = 1$, then $y'(1) > 0$.

These results comprise the essence of the maximum principle. There are some elementary consequences of these facts that will be needed later in the paper. Consider the two-point boundary conditions

$$B_o[y] = a_o y(0) - b_o y'(0),$$

$$B_1[y] = a_1 y(1) + b_1 y'(1).$$

We assume that $a_i \geq 0$, $b_i \geq 0$, $a_o + b_o > 0$, $a_1 + b_1 > 0$, and that we do not have

$$a_o = a_1 = 0, \quad \beta(t) \equiv 0.$$

It is easy to show that if $y(t) \in C^1[0,1] \cap C^2(0,1)$ satisfies inequality (2.1) and the conditions

$$B_o[y] \leq 0, \; B_1[y] \leq 0,$$

then $y(t) \leq 0$ for $0 \leq t \leq 1$. From this, we see that if $y(t)$ satisfies (2.1), then

$$y(t) \leq \max\,(0, y(0), y(1)) \quad (0 \leq t \leq 1).$$

Similar results can be proved if $Ly(t) \leq 0$ by replacing y by $-y$.

We now consider the specific boundary value problem

$$(2.2) \qquad \begin{cases} Ly(t) = 0 & (0 < t < 1) \\ y(0) = A, \; y(1) = B & , \end{cases}$$

where $\alpha(t)$ and $\beta(t)$ are assumed to be continuous. There is a unique solution to the problem (2.2) [1 , p. 96], and from the maximum principle it follows that

$$\min(0,A,B) \le y(t) \le \max(0,A,B) \qquad (0 \le t \le 1).$$

Furthermore, if $\beta(t) \equiv 0$ we have

$$\min(A,B) \le y(t) \le \max(A,B) \qquad (0 \le t \le 1).$$

Later in the paper, we will need conditions which ensure that either $y(t)$ or $y'(t)$ has one sign for $0 \le t \le 1$. The following sufficient conditions are easy consequences of the above remarks:

(a) If $AB \ge 0$, then $y(t)$ has one sign.

(b) If $\beta(t) \equiv 0$, then $y'(t)$ has one sign.

(c) If $AB \le 0$, then $y'(t)$ has one sign.

In the remainder of this section, we will derive some a-priori bounds and existence theorems for solutions to quasilinear problems of the form

$$(2.3) \qquad \begin{cases} \epsilon y'' + \alpha(t,y,\epsilon)y' - \beta(t,y,\epsilon)y = \gamma(t,y,\epsilon) & (a < t < b) \\ y(a) = A(\epsilon),\ y(b) = B(\epsilon). \end{cases}$$

We assume that α, β, and γ are continuous functions and that $\beta \ge 0$. As an immediate consequence of the maximum principle, it is easy to see that if $y(t)$ is a solution of (2.3) with

$\gamma(t,y,\epsilon) \equiv 0$, then

$$\max_{0 \le t \le 1} |y(t)| \le \max(|A(\epsilon)|, |B(\epsilon)|).$$

With the use of a suitable comparison function, this result can easily be extended to include the case $\gamma(t,y,\epsilon) \not\equiv 0$. Assume that there is a continuous function $\alpha_o(t,\epsilon)$ such that

$$(2.4) \qquad |\alpha(t,y,\epsilon)| \le \alpha_o(t,\epsilon).$$

Then using the construction developed in [25 , Sec. 1.5], it is easy to show that there exists a constant $M(\epsilon)$ such that

$$(2.5) \quad \max_{0 \le t \le 1} |y(t)| \le \max(|A(\epsilon)|, |B(\epsilon)|) + M(\epsilon) \max_{0 \le t \le 1} |\gamma(t,y(t),\epsilon)|.$$

With this estimate, we can prove the existence of a solution to the quasilinear problem.

Theorem 2.1. Assume either that $\gamma(t,y,\epsilon) \equiv 0$ or that (2.4) is satisfied. Also assume that there exists a continuous function $\gamma_o(t,\epsilon)$ such that

$$(2.6) \qquad |\gamma(t,y,\epsilon)| \le \gamma_o(t,\epsilon).$$

Then there exists a solution $y(t)$ to the problem (2.3).

Proof. Define an operator T: $\bar{y}(t) \rightarrow y(t)$, where $y(t)$ is the solution of

$$\begin{cases} \epsilon y'' + \alpha(t,\bar{y},\epsilon)y' - \beta(t,\bar{y},\epsilon)y = \gamma(t,\bar{y},\epsilon) & (a < t < b) \\ y(a) = A(\epsilon),\ y(b) = B(\epsilon). \end{cases}$$

A standard argument using (2.5) and (2.6) can be used to show that the operator T has a fixed point $y(t)$, which is then a solution of (2.3). (cf. [9 , Thm. 1]).

Remark. The condition in (2.6) can be relaxed to only require that $\gamma(t,y,\epsilon) = \gamma_1(t,y,\epsilon) + \gamma_2(t,y,\epsilon)$, where $\frac{\partial\gamma_1}{\partial y} \geq 0$ and $|\gamma_2(t,y,\epsilon)| \leq \gamma_o(t,\epsilon)$. To see this, we write

$$\gamma(t,y,\epsilon) = \gamma_1(t,0,\epsilon) + y\,\frac{\partial\gamma_1(t,\xi,\epsilon)}{\partial y} + \gamma_2(t,y,\epsilon).$$

The existence of a solution to (2.3) then follows by applying Theorem 2.1 to the differential operator

$$Ly = \epsilon y'' + \alpha(t,y,\epsilon)\,y' - [\beta(t,y,\epsilon) + \frac{\partial\gamma_1(t,\xi,\epsilon)}{\partial y}]y\ .$$

For the general quasilinear problem, we frequently have $\lim_{\epsilon\to 0+} M(\epsilon) = +\infty$ in (2.5). The next two theorems give sufficient conditions for having $M(\epsilon)$ bounded independent of ϵ.

Theorem 2.2. Let $y(t)$ be a solution of (2.3) and assume that:

(a) $\alpha(t,y(t),\epsilon)$ has one sign,

(b) $|\alpha(t,y(t),\epsilon)| + \beta(t,y(t),\epsilon) \geq \alpha_o > 0$.

Then (2.5) is satisfied with

$$M(\epsilon) = \frac{1}{\alpha_0}\,[(b-a)(b-a+1) + 1].$$

<u>Proof</u>. If $v(t)$ is a continuous function, define

$$\|v(t)\|_\infty = \max_{a\le t\le b} |v(t)|.$$

We first consider the case $\alpha(t,y,\epsilon) \le 0$. Define a comparison function $\phi(t)$ by

$$\phi(t) = \frac{1}{\alpha_0}\,[t^2 - (1 + 2b)t + (2ab + a - a^2 - 1)].$$

If we let

$$w(t) = y(t) + \|\gamma(t,y(t),\epsilon)\|_\infty\,\phi(t),$$

it follows that

$$\epsilon w''(t) + \alpha(t,y(t),\epsilon)w'(t) - \beta(t,y(t),\epsilon)\,w(t) \ge 0 \quad (a < t < b).$$

With the use of the maximum principle, we then have

$$\text{(2.7)} \qquad y(t) \le \max(|A(\epsilon)|, |B(\epsilon)|) - \|\gamma(t,y(t),\epsilon)\|_\infty\,\phi(b).$$

A similar argument with the use of the function

$$w(t) = -\,y(t) + \|\gamma(t,y(t),\epsilon)\|_\infty\,\phi(t)$$

yields (2.7) with $y(t)$ replaced by $-y(t)$ on the left-hand side of the inequality. This completes the proof for the case

$\alpha(t,y,\epsilon) \le 0$. If $\alpha(t,y,\epsilon) \ge 0$, the proof follows in the same fashion with the use of the comparison function

$$\phi(t) = \frac{1}{\alpha_0}\,[t^2 + (1 - 2a)t + (2ab - b - b^2 - 1)].$$

<u>Theorem 2.3</u>. Let $y(t)$ be a solution of (2.3) and assume that

$$\beta(t, y(t), \epsilon) \ge \beta_0 > 0.$$

Then (2.5) is satisfied with $M(\epsilon) = \frac{1}{\beta_0}$.

<u>Proof.</u> The proof follows as in Theorem 2.2 with the use of the comparison function $\phi(t) = -\frac{1}{\beta_0}$.

Comparison functions can also be used to determine the <u>rate of decay</u> of $y(t)$ as $\epsilon \to 0+$. The next three results give examples of this technique.

<u>Theorem 2.4</u>. Let $y(t) = y(t,\epsilon)$ be a solution of (2.3) with $\gamma(t,y,\epsilon) \equiv 0$, and assume that:

(a) There is a function $\alpha_0(t) \in C^1[a,b]$ such that

$$\alpha(t,y,\epsilon) \ge \alpha_0(t) \ge 0 \text{ and } \beta(t,y,\epsilon) \ge -\alpha_0'(t).$$

(b) Either $B(\epsilon) \equiv 0$ or $\beta(t,y,\epsilon) \equiv 0$.

If we set $\alpha_1(t) = \int_a^t \alpha_0(x)dx$, then

$$|y(t,\epsilon) - B(\epsilon)| \leq |A(\epsilon) - B(\epsilon)| \exp\left[-\frac{1}{\epsilon}\alpha_1(t)\right] \qquad (a \leq t \leq b).$$

Proof. Define a function $\emptyset(t,\epsilon)$ by

$$\emptyset(t,\epsilon) = (y(t,\epsilon) - B(\epsilon)) \exp\left[\frac{1}{\epsilon}\alpha_1(t)\right] \quad (a \leq t \leq b).$$

It is easy to verify that $\emptyset(t) = \emptyset(t,\epsilon)$ satisfies

$$\epsilon\emptyset'' + (\alpha - 2\alpha_0)\emptyset' + \left(\frac{\alpha_0}{\epsilon}(\alpha_0 - \alpha) - \alpha_0' - \beta\right)\emptyset = 0.$$

By assumption we have

$$\frac{\alpha_0}{\epsilon}(\alpha_0 - \alpha) - \alpha_0' - \beta \leq 0,$$

and hence

$$|\emptyset(t,\epsilon)| \leq |A(\epsilon) - B(\epsilon)|.$$

The theorem then follows easily from this fact.

We can also give an analog of Theorem 2.4 for the case $\alpha(t,y,\epsilon) \leq 0$.

Theorem 2.5. Let $y(t) = y(t,\epsilon)$ be a solution of (2.3) with $\gamma(t,y,\epsilon) \equiv 0$, and assume that:

(a) There is a function $\alpha_0(t) \in C^1[a,b]$ such that $\alpha(t,y,\epsilon) \leq \alpha_0(t) \leq 0$ and $\beta(t,y,\epsilon) \geq -\alpha_0'(t)$.

(b) Either $A(\epsilon) \equiv 0$ or $\beta(t,y,\epsilon) \equiv 0$.

If we set $\alpha_1(t) = \int_b^t \alpha_0(x)dx$, then

$$|y(t,\epsilon) - A(\epsilon)| \le |B(\epsilon) - A(\epsilon)| \exp\left[-\frac{1}{\epsilon}\alpha_1(t)\right] \quad (a \le t \le b).$$

Proof. The proof follows as in Theorem 2.4 with the use of the function

$$\phi(t,\epsilon) = (y(t,\epsilon) - A(\epsilon)) \exp\left[\frac{1}{\epsilon}\alpha_1(t)\right].$$

The next theorem treats the case where $\alpha(t,y,\epsilon) \equiv 0$ but $\beta(t,y,\epsilon)$ is bounded from below.

Theorem 2.6. Let $y(t) = y(t,\epsilon)$ be a solution of (2.3) with $\alpha(t,y,\epsilon) \equiv 0$ and $\gamma(t,y,\epsilon) \equiv 0$, and assume that:

(a) There is a nonnegative function $\beta_0(t) \in C^1[a,b]$ such that

$$\beta(t,y,\epsilon) \ge [\beta_0^2(t) + \sqrt{\epsilon}\,|\beta_0'(t)|].$$

(b) Either $A(\epsilon) \equiv 0$ or $B(\epsilon) \equiv 0$.

If $A(\epsilon) \equiv 0$ we let $\beta_1(t) = \int_t^b \beta_0(x)dx$, and then

$$|y(t,\epsilon)| \le |B(\epsilon)| \exp\left[-\frac{1}{\sqrt{\epsilon}}\beta_1(t)\right] \quad (a \le t \le b).$$

If $B(\epsilon) \equiv 0$ we let $\beta_2(t) = \int_a^t \beta_0(x)dx$, and then

$$|y(t,\epsilon)| \le |A(\epsilon)| \exp\left[-\frac{1}{\sqrt{\epsilon}}\beta_2(t)\right] \quad (a \le t \le b).$$

Proof. The proof follows as in Theorems 2.4 and 2.5 with the use of the function

$$\phi(t,\epsilon) = \begin{cases} y(t,\epsilon)\ \exp\left[\frac{1}{\sqrt{\epsilon}}\ \beta_1(t)\right] & \text{if } A(\epsilon) \equiv 0 \\ y(t,\epsilon)\ \exp\left[\frac{1}{\sqrt{\epsilon}}\ \beta_2(t)\right] & \text{if } B(\epsilon) \equiv 0. \end{cases}$$

We now consider a-priori bounds for $y'(t)$. If $\alpha(t,y,\epsilon) \geq \alpha_0 > 0$ we expect in general to have a boundary layer at $t = a$, so the bound for $y'(t)$ cannot be uniform in the whole interval $[a,b]$. However, it is uniform in any interval $[a + \delta, b]$ for $0 < \delta < b - a$.

Theorem 2.7. Let $y(t)$ be a solution of (2.3), and assume that $\alpha(t,y,\epsilon) \geq \alpha_0 > 0$. Define

$$\gamma_0 = \max_{a \leq t \leq b} |\gamma(t,y(t),\epsilon)|,$$

$$\beta_0 = \max\left(2,\ (b-a)\left[\frac{1}{(b-a)(b-a+1)+1} + \max_{a \leq t \leq b} \frac{\beta(t,y(t),\epsilon)}{\alpha(t,y(t),\epsilon)}\right]\right),$$

and

$$M = \beta_0\left[\max(|A|,|B|) + \frac{\gamma_0}{\alpha_0}\ ((b-a)(b-a+1) + 1)\right].$$

If we have $a \leq t < t_0 \leq b$, then

$$(2.8) \qquad |y(t) - y(t_0)| \leq M \left|\frac{t-t_0}{a-t_0}\right| ,$$

and hence

$$(2.9) \qquad |y'(t)| \le \frac{M}{t-a} \qquad (a < t \le b)$$

Proof. Define a comparison function

$$\phi(t) = y(t) - y(t_0) + M\left(\frac{t-t_0}{a-t_0}\right).$$

Then $\phi(t)$ satisfies

$$\begin{cases} \epsilon\phi'' + \alpha(t,y,\epsilon)\phi' - \beta(t,y,\epsilon)\phi \le 0 & (a < t < t_0), \\ \phi(a) \ge 0,\ \phi(t_0) = 0 & , \end{cases}$$

and hence

$$y(t) - y(t_0) \ge -M\left(\frac{t-t_0}{a-t_0}\right) \qquad (a \le t \le t_0).$$

A similar argument with the use of the comparison function

$$\phi(t) = y(t) - y(t_0) - M\left(\frac{t-t_0}{a-t_0}\right)$$

proves that

$$y(t) - y(t_0) \le M\left(\frac{t-t_0}{a-t_0}\right) \qquad (a \le t \le t_0).$$

This proves (2.8), which easily implies (2.9) by taking the limit as t tends to t_0.

It is easy to obtain a bound for $y'(a)$, although this bound may become infinite as $\epsilon \to 0+$. Let $y(t) = y(t,\epsilon)$ be a solution of

(2.3) with $\alpha(t,y,\epsilon) \geq \alpha_0 > 0$, and assume that there is a constant M such that

$$|y(t,\epsilon)| \leq M \qquad (a \leq t \leq b). \tag{2.10}$$

Then there exists a positive constant M_1 such that

$$|y'(a,\epsilon)| \leq \frac{M_1}{\epsilon} \qquad (0 < \epsilon \leq \epsilon_0). \tag{2.11}$$

To prove this, we use an integrating factor to write (2.3) in the form

$$\epsilon(y'(t)e^{Q(t)})' = (\bar{\beta}(t)y(t) + \bar{\gamma}(t))e^{Q(t)}, \tag{2.12}$$

where $\bar{\beta}(t) = \beta(t,y(t),\epsilon)$, $\bar{\gamma}(t) = \gamma(t,y(t),\epsilon)$, and

$$Q(t) = \frac{1}{\epsilon}\int_a^t \alpha(\tau,y(\tau),\epsilon)d\tau.$$

By integrating eq. (2.12), we see that

$$y'(t) = e^{-Q(t)}\left[\frac{1}{\epsilon}\int_a^t (\bar{\beta}(s)y(s) + \bar{\gamma}(s))e^{Q(s)}\, ds + \left(\frac{B-A-R(b)}{\int_a^b e^{-Q(s)}ds}\right)\right]; \tag{2.13}$$

where

$$R(t) = \frac{1}{\epsilon}\int_a^t \int_a^x (\bar{\beta}(s)y(s) + \bar{\gamma}(s))e^{Q(s)-Q(x)}\, ds\, dx\ .$$

Then (2.11) follows easily from (2.10) - (2.13).

This construction can also be used to establish a uniform bound for $y'(t)$ in the case where the boundary condition at $t = 0$ is changed.

Theorem 2.8. Consider the problem

$$(2.14) \qquad \begin{cases} \epsilon y'' + \alpha(t,y,\epsilon)y' - \beta(t,y,\epsilon)y = \gamma(t,y,\epsilon) & (a<t<b), \\ y'(a) = A, \quad a_1 y(1) + b_1 y'(1) = B, \end{cases}$$

where $\alpha(t,y,\epsilon) \geq \alpha_0 > 0$, $a_1 \geq 0$, $b_1 \geq 0$,

$a_1 + b_1 > 0$, and if $\beta(t,y,\epsilon) \equiv 0$, then $a_1 > 0$. Assume that there exists a solution $y(t) = y(t,\epsilon)$ of (2.14), and a constant M such that (2.10) holds. Then there exists a positive constant M_1 such that

$$|y'(t)| \leq M_1 \qquad (a \leq t \leq b).$$

Proof. As before, we integrate eq. (2.12) to find that

$$y'(t) = Ae^{-Q(t)} + \frac{1}{\epsilon}\int_a^t (\bar{\beta}(s)y(s) + \bar{\gamma}(s))e^{Q(s)-Q(t)}ds.$$

Thus

$$|y'(t)| \leq |A| + \frac{M_0}{\epsilon}\int_a^t e^{Q(s)-Q(t)} \, ds$$

$$\leq \quad |A| + \frac{M_0}{\epsilon} \int_a^t e^{\frac{\alpha_0}{\epsilon}(s-t)} \, ds$$

$$\leq \quad |A| + \frac{M_0}{\alpha_0} \equiv M_1 \, .$$

3. Applications to Linear Problems

We now consider some applications of the results in the preceding section to linear problems of the form

$$(3.1)\quad \begin{cases} \epsilon y''(t) + \alpha_1(t,\epsilon)y'(t) - \alpha_2(t,\epsilon)y(t) = \alpha_3(t,\epsilon) & (a < t < b) \\ y(a) = A(\epsilon), \quad y(b) = B(\epsilon) . \end{cases}$$

We assume that $\lim_{\epsilon \to 0+} A(\epsilon) = A(0)$ and $\lim_{\epsilon \to 0+} B(\epsilon) = B(0)$. Many of the examples we consider can be treated as special cases of existing theorems (cf. [21] and the references in that paper). However, the proofs of these results are very simple with the use of the maximum principle, and do not require strong assumptions on the differentiability of the coefficient functions $\alpha_i(t,\epsilon)$. In addition, Theorems 2.4 - 2.6 can be used to give very simple estimates on the rate of convergence as $\epsilon \to 0+$.

We first define a set of functions D_0 which will be used to specify the smoothness of the coefficient functions $\alpha_i(t,\epsilon)$. If $\gamma(t,\epsilon)$ is a given sequence of functions defined for $0 < \epsilon \leq \epsilon_0$, then $\gamma(t,\epsilon) \in D_0$ if there exists a function $\bar{\gamma}(t)$ such that $\gamma(t,\epsilon) \in C^1[a,b]$, $\gamma(t,\epsilon)$ converges uniformly to $\bar{\gamma}(t)$ as $\epsilon \to 0+$, and $|\gamma'(t,\epsilon)| \leq M_1$ for $a \leq t \leq b$ and $0 < \epsilon \leq \epsilon_0$. We assume that the coefficient functions $\alpha_i(t,\epsilon)$ in (3.1) satisfy:

(a) $\alpha_i(t, \epsilon) \in D_0$ for $1 \leq i \leq 3$,

(b) $\alpha_2(t,\epsilon) \geq 0$.

<u>Theorem 3.1.</u> Let $y(t,\epsilon)$ be the solution of (3.1), and assume that $|\alpha_1(t,\epsilon)| \geq \delta > 0$. If $\alpha_1(t,\epsilon) < 0$, let $y_0(t)$ be the solution to the initial value problem

$$\begin{cases} \bar{\alpha}_1(t)\, y_0'(t) - \bar{\alpha}_2(t)\, y_0(t) = \bar{\alpha}_3(t) & (a < t < b) \\ y_0(a) = A(0) \ , \end{cases}$$

and if $\alpha_1(t,\epsilon) > 0$, let $y_0(t)$ be the solution to the terminal value problem

$$\begin{cases} \bar{\alpha}_1(t)\, y_0'(t) - \bar{\alpha}_2(t)\, y_0(t) = \bar{\alpha}_3(t) & (a < t < b) \\ y_0(b) = B(0) \ . \end{cases}$$

Then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = y_0(t) \qquad (a < t < b).$$

<u>Proof.</u> We give the proof for the case $\alpha_1(t,\epsilon) < 0$, and the proof for the case $\alpha_1(t,\epsilon) > 0$ follows in a similar fashion. Let $\emptyset(t,\epsilon)$ be the solution to

$$\begin{cases} \alpha_1(t,\epsilon)\, \emptyset'(t) - \alpha_2(t,\epsilon)\, \emptyset(t) = \alpha_3(t,\epsilon) & (a < t < b) \\ \emptyset(a) = A(\epsilon) \ . \end{cases}$$

With the assumptions on $\alpha_i(t,\epsilon)$, it is easy to show that

$$|\phi''(t,\epsilon)| \leq M_2 \qquad (a \leq t \leq b, \quad 0 < \epsilon \leq \epsilon_0) .$$

Furthermore, from a standard theorem on the continuity of solutions to initial value problems [6, p. 29], it follows that

$$\lim_{\epsilon \to 0} \phi(t,\epsilon) = y_0(t) \qquad (a \leq t < b) .$$

Let $\psi(t,\epsilon)$ be the solution to

$$\begin{cases} \epsilon\psi''(t) + \alpha_1(t,\epsilon)\,\psi'(t) - \alpha_2(t,\epsilon)\psi(t) = \alpha_3(t,\epsilon) + \epsilon\phi''(t,\epsilon) & (a<t<b) \\ \psi(a) = A(\epsilon),\ \psi(b) = B(\epsilon) . \end{cases}$$

From Theorem 2.2 we have

$$\| y(t,\epsilon) - \psi(t,\epsilon)\|_\infty \leq C\ \|\epsilon\phi''(t,\epsilon)\|_\infty \leq \epsilon\, C M_2 .$$

Finally, from Theorem 2.5 it follows that

$$\begin{aligned} |\psi(t,\epsilon) - \phi(t,\epsilon)| &\leq |B(\epsilon) - \phi(b,\epsilon)|\ \exp\left[-\frac{\delta}{\epsilon}\ (b-t)\right] \\ &\leq M_3 \exp\left[-\frac{\delta}{\epsilon}\ (b-t)\right] \end{aligned}$$

for $0 < \epsilon \leq \epsilon_0$. The theorem now follows easily from these results.

Using a similar technique with a different comparison theorem, we can also treat the case $\alpha_1(t,\epsilon) \equiv 0$.

Theorem 3.2. Let $y(t,\epsilon)$ be the solution of (3.1), and assume that $\alpha_1(t,\epsilon) \equiv 0$ and $\alpha_2(t,\epsilon) \geq \delta > 0$. Then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = -\frac{\bar{\alpha}_3(t)}{\bar{\alpha}_2(t)} \qquad (a < t < b).$$

Proof. Let $w(t,\epsilon) = -\dfrac{\alpha_3(t,\epsilon)}{\alpha_2(t,\epsilon)}$, and define three auxiliary functions $\phi_i(t,\epsilon)$ as the solutions of:

$$\begin{cases} \epsilon\phi_1'' - \alpha_2\phi_1 = -\epsilon w'' & (a < t < b) \\ \phi_1(a) = \phi_1(b) = 0, & \end{cases}$$

$$\begin{cases} \epsilon\phi_2'' - \alpha_2\phi_2 = 0 & (a < t < b) \\ \phi_2(a) = A(\epsilon) - w(a), \quad \phi_2(b) = 0, & \end{cases}$$

$$\begin{cases} \epsilon\phi_3'' - \alpha_2\phi_3 = 0 & (a < t < b) \\ \phi_3(a) = 0, \quad \phi_3(b) = B(\epsilon) - w(b). & \end{cases}$$

Thus we have

$$y(t,\epsilon) - w(t,\epsilon) = \phi_1(t,\epsilon) + \phi_2(t,\epsilon) + \phi_3(t,\epsilon) \qquad (a \leq t \leq b).$$

With the assumptions on $\alpha_i(t,\epsilon)$, we see that $|w''(t,\epsilon)| \leq M$. From Theorem 2.3 we then have

$$|\phi_1(t,\epsilon)| \leq \frac{M\epsilon}{\delta}.$$

If we apply Theorem 2.6 with $\beta_0(t) = \sqrt{\delta}$, we see that

$$|\phi_2(t,\epsilon)| \leq |A(\epsilon) - w(a,\epsilon)| \exp\left[-(t-a)\sqrt{\frac{\delta}{\epsilon}}\,\right],$$

$$|\phi_3(t,\epsilon)| \leq |B(\epsilon) - w(b,\epsilon)| \exp\left[-(b-t)\sqrt{\frac{\delta}{\epsilon}}\,\right].$$

The theorem now follows easily from these results.

Remark. Carrier [3 , pp. 176-177] gives the following example, which can be treated by Theorem 3.2:

$$\begin{cases} \epsilon y''(t,\epsilon) - (2 - t^2)\, y(t,\epsilon) = -1 & (-1 < t < 1) \\ y(-1) = y(1) = 0 . \end{cases}$$

We now consider the linear problem

$$\text{(3.2)} \qquad \begin{cases} \epsilon y''(t) + \alpha(t,\epsilon)\, y'(t) - \beta(t,\epsilon)\, y(t) = 0 & (a < t < b) \\ y(a) = A(\epsilon),\ y(b) = B(\epsilon) , \end{cases}$$

where:

(a) $\alpha(t,\epsilon) \in D_0$, $\beta(t,\epsilon) \in D_0$, and $\beta(t,\epsilon) \geq 0$,

(b) $\lim_{\epsilon \to 0+} A(\epsilon) = A(0)$ and $\lim_{\epsilon \to 0+} B(\epsilon) = B(0)$.

First, we recall an elementary result for problems of the form of (3.2). Define S_0 to be the set of all possible limits of $y(t,\epsilon)$

as $\epsilon \to 0+$. That is, S_0 is the set of all $Y(t) \in L^1[a,b]$ such that there exists a sequence $\epsilon_n \to 0+$ with $\lim_{n\to\infty} y(t,\epsilon_n) = Y(t)$ for $a \le t \le b$. Using the maximum principle, it is easy to show that $\{y(t,\epsilon) \,|\, 0 < \epsilon \le \epsilon_0\}$ is a uniformly bounded family of functions with uniformly bounded total variation. Thus it follows from the Helly selection theorem [17, p. 222] that S_0 is not empty. The <u>reduced equation</u> associated with (3.2) is

$$\bar{\alpha}(t)\, Y'(t) - \bar{\beta}(t)\, Y(t) = 0 \ . \tag{3.3}$$

We would expect any limit function $Y(t) \in S_0$ to satisfy (3.3), and in fact the following result is an immediate consequence of Theorem 3.1.

<u>Theorem 3.3</u>. Let $y(t,\epsilon)$ be the solution of (3.2) and let $Y(t) \in S_0$. If there is an interval $(c,d) \subset (a,b)$ such that

$$\bar{\alpha}(t) \not\equiv 0 \qquad (c < t < d) \ ,$$

then $Y(t) \in C^1(c,d)$, and $Y(t)$ satisfies (3.3) for $c < t < d$.

As an example of an application of these results, we consider the following simple turning point problem. It is clear that the result could also be extended to a function $\alpha(t,\epsilon)$ with a finite number of interior nodal zeroes.

Theorem 3.4. Let $y(t,\epsilon)$ be the solution of (3.2), and assume that there is a $\delta(\epsilon) \in (a,b)$ such that

$$(3.4) \qquad \alpha(t,\epsilon) \begin{cases} < 0 & \text{if } a \leq t < \delta(\epsilon) \\ = 0 & \text{if } t = \delta(\epsilon) \\ > 0 & \text{if } \delta(\epsilon) < t \leq b \,. \end{cases}$$

Also assume that $\lim_{\epsilon \to 0} \delta(\epsilon) = \bar{\delta}$ and $\bar{\alpha}(t) \neq 0$ if $t \in [a,b]$ and $t \neq \bar{\delta}$. Then

$$\lim_{\epsilon \to 0} y(t,\epsilon) = \begin{cases} A(0) \exp\left[\int_a^t \frac{\bar{\beta}(x)}{\bar{\alpha}(x)}\, dx\right] & \text{if } a \leq t < \bar{\delta} \\ \\ B(0) \exp\left[\int_b^t \frac{\bar{\beta}(x)}{\bar{\alpha}(x)}\, dx\right] & \text{if } \bar{\delta} < t \leq b. \end{cases}$$

Proof. The proof follows easily from Theorem 3.1.

In the analog of Theorem 3.4 for the quasi-linear case, we can derive a non-trivial lower bound for the solution $y(t,\epsilon)$. We first state a slightly modified form of the maximum principle.

Lemma 3.1 Let $\delta \in (a,b)$, and let T be the set of all functions $\psi(t) \in C[a,\delta) \cap C(\delta,b]$ such that $\lim_{t \to \delta-} \psi(t)$ and $\lim_{t \to \delta+} \psi(t)$ exist. Let $\phi_1(t)$ and $\phi_2(t)$ be bounded functions on $[a,b]$ with $\phi_2(t) \geq 0$. Let $w(t)$ be a function that satisfies:

(a) $w(t) \in C[a,b]$, $w'(t) \in T$, $w''(t) \in T$,

(b) $w''(t) + \emptyset_1(t)\, w'(t) - \emptyset_2(t)\, w(t) \geq 0$ for $a < t < b$, $t \neq \delta$,

(c) $w(a) \leq 0$, $w(b) \leq 0$.

If there is a point $t_0 \in (a,b)$ such that $w(t_0) > 0$, then $w'(\delta^+) < 0 < w'(\delta^-)$.

Proof. If $w(t_0) > 0$, it follows from the maximum principle applied to $w(t)$ on the intervals (a,δ) and (δ,b) that

$$\max_{a \leq t \leq \delta} w(t) = \max_{\delta \leq t \leq b} w(t) = w(\delta) > 0 ,$$

and hence $w'(\delta^+) < 0 < w'(\delta^-)$.

Theorem 3.5. Let $y(t,\epsilon)$ be a solution of (2.3) with $\gamma(t,y,\epsilon) \equiv 0$, $A(\epsilon) > 0$, and $B(\epsilon) > 0$. Assume that there exists a $\delta = \delta(\epsilon) \in (a,b)$ and a constant $M > 0$ such that

$$\alpha(t,y(t),\epsilon) \begin{cases} < 0 & \text{if } a < t < \delta \\ = 0 & \text{if } t = \delta \\ > 0 & \text{if } \delta < t < b , \end{cases}$$

and

$$\left| \frac{\beta(t,y(t),\epsilon)}{\alpha(t,y(t),\epsilon)} \right| \leq M \qquad (a < t < b,\ t \neq \delta) .$$

Then

$$y(t,\epsilon) \geq \min\,(A(\epsilon),\; B(\epsilon))\;\; \exp\;[-\,M(b-a)]\,.$$

Proof. Let $\mu = \mu(\epsilon)$ be the solution to

$$A(\epsilon)\; e^{M(b+a-2\mu)} = B(\epsilon)\;.$$

Then we have two cases to consider:

Case 1. $\mu \leq \delta$. Define a function $\phi(t)$ by

$$\phi(t) \quad = \quad \begin{cases} A(\epsilon)\;\exp\,[-\,M(t-a)] & \text{if } \;a \leq t \leq \delta \\ \\ A(\epsilon)\;\exp\,[\,M(t-2\delta+a)] & \text{if } \;\delta \leq t \leq b\;. \end{cases}$$

If we apply Lemma 3.1 to the function $w(t) = \phi(t) - y(t)$, we have

$$y(t,\epsilon) \geq \phi(t) \geq A(\epsilon)\;\exp\;[-M(b-a)\,]\;.$$

Case 2. $\mu \geq \delta$. As in case 1, we define a comparison function

$$\phi(t) \quad = \quad \begin{cases} B(\epsilon)\;\exp\,[-\,M(t-2\delta+b)\,] & \text{if } \;a \leq t \leq \delta \\ B(\epsilon)\;\exp\,[\,M(t-b)\,] & \text{if } \;\delta \leq t \leq b\;. \end{cases}$$

It is then easy to show that

$$y(t,\epsilon) \geq \phi(t) \geq B(\epsilon) \exp[-M(b-a)] ,$$

and this completes the proof of the theorem.

We can also consider a problem similar to (3.2) in which the sign conditions on $\alpha(t,\epsilon)$ in (3.4) are reversed. The following result shows that we cannot have a result such as Theorem 3.5 for this case, since the limit function is identically zero.

Theorem 3.6 Let $y(t,\epsilon)$ be the solution of (3.2) with $A(\epsilon) \geq 0$ and $B(\epsilon) \geq 0$, and assume that there exists a $\delta_0 \in (0, b-a)$ such that

$$\bar{\alpha}(a+t) > 0 ,$$

$$\bar{\alpha}(b-t) < 0 ,$$

and

$$\bar{\beta}(a+t)\,\bar{\beta}(b-t) > 0$$

for $0 < t \leq \delta_0$. Then

$$\lim_{\epsilon \to 0} y(t,\epsilon) = 0 \qquad (a < t < b) .$$

Proof. If $Y(t) \in S_0$, it follows from Theorem 3.3 that $Y(t)$ is continuously differentiable and satisfies (3.3) on the intervals $(a, a+\delta_0)$ and $(b-\delta_0, b)$. Suppose that we do not have $Y(t) \equiv 0$ for $a < t < b$. We then have two cases to consider:

Case 1. $Y(t) = 0$ for all $t \in (a, a + \delta_0)$. Then there exists a point $t_0 \in [a + \delta_0, b)$ such that $Y(t_0) > 0$. From the maximum principle, we have $Y(t) \geq Y(t_0) > 0$ for $t_0 \leq t < b$. Equation (3.3) then implies that there exists a point $t_1 \in [t_0, b) \cap (b - \delta_0, b)$ such that $Y'(t_1) < 0$, and this contradicts the maximum principle.

Case 2. $Y(t_0) > 0$ for some $t_0 \in (a, a + \delta_0)$. Then from (3.3) we have $Y'(t_0) > 0$, and so $Y(t) \geq Y(t_0) > 0$ for $t_0 \leq t < b$. As in case 1, this implies that there exists a point $t_1 \in [t_0, b) \cap (b - \delta_0, b)$ such that $Y'(t_1) < 0$. Because of the maximum principle this cannot happen, and hence $Y(t) \equiv 0$ for $a < t < b$. The proof of the theorem can then be completed by using a standard argument on the uniqueness of the limit function $Y(t)$.

4. Applications to Quasilinear Problems

Consider the quasilinear boundary value problem

$$(4.1) \qquad \begin{cases} \epsilon y'' + \alpha(t,y,\epsilon)y' = \gamma(t,y,\epsilon) & (a < t < b) \\ y(a) = A(\epsilon), \quad y(b) = B(\epsilon) \end{cases}$$

where the functions $\alpha(t,y,\epsilon)$, $\gamma(t,y,\epsilon)$ are uniformly continuous, and continuously differentiable with respect to (t,y) in any region of the form

$$(4.2) \qquad \mathbb{R}(k) \equiv \{(t,y,\epsilon) \mid a \le t \le b, \ |y| \le k, \ 0 \le \epsilon \le 1\},$$

and the functions $A(\epsilon)$, $B(\epsilon)$ are uniformly continuous for $0 \le \epsilon \le 1$.

In this section we study these problems with the aid of the estimates of Section 2. Our first result complements the work of Coddington and Levinson [5].

Let

$$(4.3) \qquad \begin{cases} \bar{A} \equiv \max \{|A(\epsilon)| \mid 0 \le \epsilon \le 1\} \\ \bar{B} \equiv \max \{|B(\epsilon)| \mid 0 \le \epsilon \le 1\} \end{cases}$$

<u>Theorem 4.1</u>: Suppose there are three positive constants ϵ_0, M, α_0 such that

(i) For $0 < \epsilon \le \epsilon_0$ there is a solution $y(t,\epsilon)$ of the problem (4.1).

(ii) $|\gamma(t,y(t,\epsilon),\epsilon)| \le M$.

(iii) $0 < \alpha_0 \le \alpha(t,y(t,\epsilon),\epsilon)$.

Then, there is a function $u(t)$ which satisfies the reduced equation

$$(4.4) \qquad \begin{cases} \alpha(t,u(t),0)\,u'(t) = \gamma(t,u(t),0) & (a \leq t \leq b) \\ u(b) = B(0) \; . \end{cases}$$

Moreover, $u(t)$ is the only solution of (4.4) which also satisfies

$$(4.5) \qquad |u(t)| \leq k_0 \equiv \mathrm{Max}\,(\bar{A},\bar{B}) + \frac{M}{\alpha_0}\,[(b-a)(b-a+1)+1].$$

Finally, for any δ, $0 < \delta < b - a$,

$$(4.6) \qquad \lim_{\epsilon \to 0+} \; \max_{a+\delta \leq t \leq b} \{|y(t,\epsilon) - u'(t)|\} = 0$$

Proof: Using Theorems 2.2 and 2.7 we see that there is a constant $M_1 > 0$ such that

$$(4.7a) \qquad |y(t,\epsilon)| \leq k_0 \qquad (a \leq t \leq b)$$

$$(4.7b) \qquad |y'(t,\epsilon)| \leq \frac{M_1}{(t-a)} \qquad (a < t \leq b) \; .$$

Thus, on the interval $[a + \delta, b]$ the functions $\{y(t,\epsilon)\}$ are uniformly bounded and equicontinuous. Hence, we can select a sequence $\epsilon_n \to 0^+$ so that the functions $y(t,\epsilon_n)$ converge uniformly to a continuous function $u(t)$. This function is absolutely continuous. Multiply (4.1) by a function $\varphi(t) \in C_0^1[0,1]$, divide by $\alpha(t,y(t,\epsilon_n),\epsilon_n)$ and integrate over the interval $[0,1]$. Then

$$\epsilon \int_0^1 \frac{\varphi(t) y'(t,\epsilon_n)\,dt}{\alpha(t,y(t,\epsilon_n),\epsilon_n)} = -\epsilon_n \int_0^1 \left[\frac{\varphi(t)}{\alpha(t,y(t,\epsilon_n),\epsilon_n)}\right]^1 y'(t,\epsilon_n)dt \to 0$$

as $\epsilon_n \to 0$. Thus, $u(t)$ is a weak solution of (4.4). A theorem of Friedrichs [13] (see [9] and the remark therein) can be applied to show that $u(t)$ is a differentiable solution of (4.4). Since (4.7a) holds, we see that (4.5) holds. We may modify the coefficients $\alpha(t,y,0)$, $\gamma(t,y,0)$ for $|y| > k_0$ and arrive at another problem which possesses only one solution. Thus, $u(t)$ is the only solution of (4.4) which satisfies (4.5).

The proof of (4.7b) follows from an argument in [5 , p. 76]. Let

(4.8a) $\qquad z(t,\epsilon) = y(t,\epsilon) - u(t).$

Then

(4.8b) $\qquad \epsilon z'' + \alpha(t,y(t,\epsilon),\epsilon)\, z' = r$

where

(4.8c) $\qquad |r| \leq k(\delta)\,[|z| + \epsilon] \qquad\qquad (a + \delta \leq t \leq b)$.

for some constant $k(\delta)$ depending on δ. Let

$$a + \delta < a + \delta' \leq t \leq b.$$

Then, integrating (4.8b) we obtain (see [6 , eq. 12])

$$|z'(t)| \leq |z'(a+\delta)| \exp\{-\frac{\alpha_0}{\epsilon}(t-a-\delta)\}$$
$$+ \frac{1}{\epsilon} \max\{|r| \,\big|\, a+\delta \leq t \leq 1\} \cdot \int_{a+\delta}^{t} e^{-\frac{\alpha_0}{\epsilon}(t-s)} ds$$

and (4.7b) follows from (4.7a).

Remark: In [5] Coddington and Levinson assumed that there is a function $u(t)$ satisfying (4.4). They then proved the existence of an ϵ_0 such that (4.1) has a solution $y(t,\epsilon)$ for $0 < \epsilon \leq \epsilon_0$. Moreover, $y(t,\epsilon)$ is unique in the sense that it is the only solution of (4.1) in a sufficiently small neighborhood of $u(t)$. Earlier, R. V. Mises [16] had studied this problem under the assumption that both $y(t,\epsilon)$ and $u(t)$ exist.

We now turn our attention to an example which has often been considered a "model" quasilinear example for singular perturbations — see [4], [7, pp. 29-38], and [20]. Consider the problem

$$(4.9) \qquad \begin{cases} \epsilon y'' + y y' - y = 0 & (0 \leq t \leq 1) \\ \quad y(0) = A(\epsilon), \; y(1) = B(\epsilon). \end{cases}$$

Lemma 4.1. Let $y(t,\epsilon)$ be a solution of (4.9). Then

$$(4.10) \qquad \min\,(A(\epsilon),\, B(\epsilon) - 1) \leq y(t,\epsilon) - t \leq \max\,(A(\epsilon),\, B(\epsilon) - 1).$$

Moreover, if

(4.11) $\quad A(\epsilon) \geq B(\epsilon) - 1$,

then $y(t,\epsilon)$ is the unique solution of (4.9).

Proof: Let

$$w(t,\epsilon) = y(t,\epsilon) - t \ .$$

Then

$$\begin{cases} \epsilon w'' + u\, w' = 0 & (0 < t < 1) \\ w(0,\epsilon) = A(\epsilon), \quad w(1,\epsilon) = B(\epsilon) - 1 & \end{cases}$$

Applying the maximum principle, we see that $w(t,\epsilon)$ is monotone and (4.10) follows at once. Moreover, if (4.11) holds, then

$$w'(t,\epsilon) = y'(t,\epsilon) - 1 \leq 0.$$

Suppose (4.11) holds and that there are two solutions, say $y(t,\epsilon)$ and $v(t,\epsilon)$. Let

$$q(t,\epsilon) = y(t,\epsilon) - v(t,\epsilon).$$

Then

$$\begin{cases} \epsilon q'' + \ y q' \ + (v' - 1)\, q = 0 & (0 < t < 1) \\ \qquad q(0,\epsilon) = q(1,\epsilon) = 0 \ . & \end{cases}$$

Thus, using the maximum principle, we see that

$$q(t,\epsilon) \equiv 0.$$

We are now able to discuss the asymptotic behavior of $y(t,\epsilon)$ in the case that $A(0)\,B(0) \geq 0$.

Theorem 4.2. Suppose $y(t,\epsilon)$ is a solution of (4.9) and

$$A(\epsilon) \geq 0, \quad B(0) > 0 \,. \tag{4.12}$$

If

$$B(0) \geq 1 \,, \tag{4.13a}$$

then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = t + B(0) - 1 \qquad (0 < t \leq 1) \,. \tag{4.13b}$$

If

$$0 < B(0) < 1, \tag{4.14a}$$

then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = \begin{cases} 0 & 0 < t \leq 1 - B(0) \,, \\ t + B(0) - 1 & 1 - B(0) \leq t \leq 1 \,. \end{cases} \tag{4.14b}$$

Proof: Let

$$\alpha \equiv \min \; (0, \; 1 - B(0) \,).$$

Then, using Lemma 4.1 we see that, for every $\delta > 0$ and ϵ sufficiently small,

$$y(t,\epsilon) \geq \delta/2 \qquad (\alpha + \delta \leq t \leq 1)$$

Applying Theorem 4.1, we have

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = t + B(0) - 1 \qquad (\alpha < t \leq 1), \tag{4.15}$$

which proves the theorem in the case when $\alpha = 0$. Suppose now that $\alpha > 0$. The maximum principle can be used to show that there is an $\epsilon_0 > 0$ such that

$$y(t,\epsilon) \geq 0 \qquad (0 \leq t \leq 1, \quad 0 < \epsilon \leq \epsilon_0). \tag{4.16}$$

Suppose the theorem is false. For each $\epsilon > 0$ and each $y(t,\epsilon)$ there is at most one point $\gamma(\epsilon)$ such that

$$\left\{\begin{array}{lll} y'(t,\epsilon) < 0 & \text{if} & 0 \leq t < \gamma(\epsilon) \\ y'(t,\epsilon) > 0 & \text{if} & \gamma(\epsilon) < t \leq 1. \end{array}\right. \tag{4.17a}$$

such a point may not exist. Consider a sequence $\epsilon_n \to 0+$ such that

$$y'(t,\epsilon) > 0 \qquad (0 \leq t \leq 1). \tag{4.17b}$$

In this case

$$0 \leq y(t,\epsilon_n) \leq y(\alpha,\epsilon_n) \qquad (0 \leq t \leq \alpha)$$

and

$$\lim_{\epsilon_n \to 0+} y(t,\epsilon_n) = 0 \qquad (0 < t \leq \alpha). \tag{4.18}$$

On the other hand, consider a sequence $\epsilon_n \to 0+$ such that (4.17a) holds. We may assume that there is a value γ_1 such that

$$\lim_{\epsilon_n \to 0+} \gamma(\epsilon_n) = \gamma_1 .$$

If $\gamma_1 = 0$, the theorem follows as in the preceding case.

Let $\rho \in (0,\gamma_1)$ and suppose

$$\lim_{\epsilon_n \to 0+} y(\rho,\epsilon_n) = \Delta > 0 ,$$

so that for ϵ_n small enough

$$y(t,\epsilon_n) \geq \Delta/2 \qquad (0 \leq t \leq \rho) .$$

Now if we apply Theorem 4.1, we see that $y(t,\epsilon)$ must converge to a function which is monotone decreasing in the interval $(0,\rho]$. However, this contradicts the choice of γ_1 and ρ, and the contradiction completes the proof of the theorem.

<u>Corollary</u>. Suppose

$$A(0) < 0, \quad B(\epsilon) \leq 0.$$

If

(4.19a) $\quad A(0) \leq -1,$

then

(4.19b) $$\lim_{\epsilon \to 0+} y(t,\epsilon) = A(0) + t \qquad (0 \leq t < 1).$$

If

(4.20a) $\quad -1 < A(0) < 0 ,$

then

$$(4.20b)\qquad \lim_{\epsilon \to 0+} y(t,\epsilon) = \begin{cases} A(0) + t & 0 \leq t \leq -A(0), \\ 0 & -A(0) \leq t < 1. \end{cases}$$

Proof: Replace $y(t,\epsilon)$ by

$$z(t,\epsilon) = y(1 - t,\epsilon).$$

Having obtained this basic result, we turn to the case when the reduced equation has "turning points". That is, $A(0)\,B(0) < 0$.

Lemma 4.2. Let $y(t,\epsilon)$ be a solution of (4.9). Suppose

$$(4.21)\qquad A(\epsilon) < 0 < B(\epsilon).$$

Then there is a unique point $C = C(y(t,\epsilon))$ such that

$$(4.22)\qquad y(C,\epsilon) = 0.$$

Moreover, if

$$(4.23a)\qquad B(\epsilon) - A(\epsilon) < 1\,,$$

then

$$(4.23b)\qquad 0 < y'(t,\epsilon) \leq 1 \qquad (0 \leq t \leq 1).$$

If

$$(4.24a)\qquad B(\epsilon) - A(\epsilon) > 1\,,$$

then

(4.24b) $y'(t,\epsilon) \geq 1$, $(0 \leq t \leq 1)$.

Proof: The existence of a point $C = C(y(t,\epsilon))$ is obvious. That C is unique follows from the maximum principle. Suppose $t_0 \in [0,C)$ and

(4.25a) $0 < y'(t_0,\epsilon) < 1$.

Then, in a neighborhood of $t = t_0$,

$$y'' = -\frac{1}{\epsilon}\, y(y' - 1) < 0 ,$$

and y' decreases as t increases. Thus

(4.25b) $0 < y'(t,\epsilon) < 1$ $(t_0 \leq t \leq C)$.

On the other hand, suppose $t_1 \in [0,C)$ and

(4.26a) $y'(t_1,\epsilon) > 1$.

The same computation shows that in the neighborhood of t_1 $y''(t,\epsilon) > 0$, and $y'(t,\epsilon)$ increases as t increases. Thus

(4.26b) $y'(t,\epsilon) > 1$ $(t_1 \leq t \leq C)$.

Finally, on the whole interval $[0,C]$, either

(4.27a) $y'(t,\epsilon) \leq 1$,

or

$$(4.27b) \qquad y'(t,\epsilon) \geq 1 .$$

Moreover, if there is strict inequality at any interior point, then there is strict inequality at $t = C$.

A similar argument on the interval $(C, 1]$ allows us to extend the result to the interval $[0, 1]$. That is, either (4.27a) holds or (4.27b) holds. Moreover, if there is strict inequality at any interior point, there is strict inequality at $t = C$.

If (4.23a) holds, then there is some interior point at which (4.27a) holds. But then (4.27a) holds on all of $[0, 1]$. Similarly, if (4.24a) holds, then there is some interior point at which (4.27b) holds and hence (4.27b) holds on all of $[0, 1]$.

<u>Remark</u>. If $B(\epsilon) - A(\epsilon) = 1$, then

$$y(t,\epsilon) = A(\epsilon) + t$$

is the unique solution of (4.9).

<u>Theorem 4.3.</u> Suppose there is an $\epsilon_0 > 0$ such that (4.21) and (4.23a) hold. Let $y(t,\epsilon)$ be a solution of (4.9). Then

$$(4.28) \qquad \lim_{\epsilon \to 0+} y(t,\epsilon) = \begin{cases} A(0) + t & 0 \leq t \leq |A(0)| , \\ 0 & |A(0)| \leq t \leq 1 - B(0), \\ B(0) - 1 + t & 1 - B(0) \leq t \leq 1. \end{cases}$$

Proof: Applying Lemma 4.2, we see that the functions $y(t,\epsilon)$ are uniformly bounded and equicontinuous. Moreover, from (4.23b) we see that

$$|A(\epsilon)| \leq C(y(t,\epsilon)) \leq 1 - B(\epsilon).$$

We can select a subsequence $\epsilon_n \to 0+$ such that

$$\lim_{\epsilon_n \to 0+} C(y(t,\epsilon_n)) = \bar{C}.$$

For every $\delta > 0$, we can extract a subsequence so that $y(\bar{C} + \delta, \epsilon_n)$ and $y(\bar{C} - \delta, \epsilon_n)$ converge. Applying Theorem 4.2 we obtain (4.28) for this subsequence. However, the uniqueness of the limit function allows us to assert the convergence of the entire family $\{y(t,\epsilon)\}$.

The case when (4.24a) holds involves a "boundary layer". As we shall see, this boundary layer may occur in the interior of the interval.

Lemma 4.3. Suppose there is an $\epsilon_0 > 0$ such that (4.21) and (4.24a) hold for $0 < \epsilon \leq \epsilon_0$. Suppose there is a sequence $\epsilon_n \to 0+$ and a constant $\bar{C}$ such that

(4.29a) $$\lim_{\epsilon_n \to 0+} C(y(t,\epsilon_n)) = \bar{C}$$

with

(4.29b) $$0 < \bar{C} < 1.$$

Then

(4.30a) $\bar{C} = \frac{1}{2}(1 - B(0) - A(0))$

and

(4.30b) $$\lim_{\epsilon_n \to 0+} y(t,\epsilon_n) = \begin{cases} A(0) + t & 0 \le t < \bar{C} \\ B(0) - 1 + t & \bar{C} < t \le 1 \end{cases}$$

Proof: Since (4.24b) holds, we have

(4.31) $1 - B(\epsilon_n) \le C(y(t,\epsilon_n)) \le |A(\epsilon_n)|$.

Again, we argue in the two intervals $[0,\bar{C}-\delta]$ and $[\bar{C}+\delta,1]$ to obtain (4.30b). Applying theorem 4.1 we see that $y'(0,\epsilon_n)$ and $y'(1,\epsilon_n)$ are uniformly bounded. Integrating (4.9) we find that

$$\frac{1}{2}\{[B(0)]^2 - [A(0)]^2\} = \int_0^1 [\lim_{\epsilon_n \to 0+} y(t,\epsilon_n)]dt = \frac{1}{2} + A(0)\,\bar{C} + (B(0)-1)(1-\bar{C}).$$

This equation can be solved for $\bar{C}$ to complete the proof.

Theorem 4.4. Suppose there is an $\epsilon_0 > 0$ such that (4.21) and (4.24a) hold for $0 < \epsilon \le \epsilon_0$. Let $\bar{C}$ be given by (4.30a). If $\bar{C} \in (0,1)$ then, as in Lemma 4.3

$$(4.32)\qquad \lim_{\epsilon\to 0+} y(t,\epsilon) = \begin{cases} A(0) + t & 0 \le t < \bar{C} \\ & \\ B(0) - 1 + t & \bar{C} < t \le 1. \end{cases}$$

If $\bar{C} \ge 1$, then

$$(4.33)\qquad \lim_{\epsilon\to 0+} y(t,\epsilon) = A(0) + t \qquad (0 < t \le 1).$$

If $\bar{C} \le 0$, then

$$(4.34)\qquad \lim_{\epsilon\to 0+} y(t,\epsilon) = B(0) - 1 + t \qquad (0 < t \le 1).$$

Proof: We can extract a sequence so that $\lim\limits_{\epsilon_n\to 0+} C(y(t,\epsilon_n)) = C_0$, a constant. If $C_0 \in (0,1)$, then we can apply Lemma 4.3 to obtain (4.32), and C_0 is given by (4.30a).

Suppose $C_0 = 1$. Then, since (4.31) holds, we have

$$(4.35)\qquad |A(0)| \ge 1 .$$

As before, we apply Theorem 4.1 in the interval $[0,1-\delta] = [0,C_0-\delta]$ to conclude that (4.33) holds. Once more $y'(0,\epsilon_n)$ remains bounded. Integrating (4.9) we find

$$\lim_{\epsilon_n\to 0+}[\epsilon_n\, y'(1,\epsilon_n)] + \frac{1}{2}\{[B(0)]^2 - (A(0)]^2\} = A(0) + \frac{1}{2} .$$

Since $y'(1,\epsilon_n) \ge 0$ we have

$$[B(0)]^2 \le [A(0) + 1]^2 .$$

Using (4.35) we obtain

$$B(0) \leq |A(0)| - 1$$

and

$$1 \leq \bar{C} . \tag{4.36}$$

Finally, suppose $C_0 = 0$. Then, since (4.31) holds, we have

$$1 - B(0) \leq 0 .$$

Arguing in a similar manner we obtain (4.34) and

$$\bar{\bar{C}} \leq 0 .$$

Since the three cases considered are mutually exclusive and exhaust all the possibilities, the theorem follows.

The case when

$$B(0) < 0 < A(0)$$

is easily studied by considering $y(1 - t, \epsilon)$. In this way we have a complete analysis of (4.9) based on Theorem 4.1 and the (frequent) use of the maximum principle.

5. Nonlinear Problems

The quasilinear equations considered in section 4 are linear in y'. In [14] Haber and Levinson (see O'Malley [23] also) consider the general nonlinear equation

$$(5.1) \qquad \begin{cases} \epsilon y'' = f(t,y,y',\epsilon) & (0 \leq t \leq 1) \\ y(0) = A, \ y(1) = B \end{cases}$$

where the "reduced" problem

$$(5.2) \qquad \begin{cases} f(t,Y(t), Y'(t),0) = 0 & (0 \leq t \leq 1) \\ Y(0) = A, \ Y(1) = B \end{cases}$$

has an "angular solution"

$$Y(t) = \begin{cases} Y_L(t) & 0 \leq t \leq t_0 \\ Y_R(t) & t_0 \leq t \leq 1 \end{cases}$$

with

$$Y_L(t_0) = Y_R(t_0), \ Y_L'(t_0) \neq Y_R'(t_0).$$

<u>Theorem</u> (Haber and Levinson). Suppose

$$\mu_1 = Y_L'(t_0) < Y_R'(t_0) = \mu_2$$

$$f_{y'}(t,Y_L(t),Y_L'(t),0) \geq k_1 > 0, \qquad (0 \leq t \leq t_0),$$

$$f_{y'}(t,Y_R(t),Y_R'(t),0) \leq -k_2 < 0, \qquad (t_0 \leq t \leq 1),$$

and

$$f(t_0,Y_L(t_0),\omega,0) > 0 \qquad (\mu_1 < \omega < \mu_2) .$$

Then, for ϵ sufficiently small, there exists a solution $y(t,\epsilon)$ of the boundary value problem (5.1) such that

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = Y(t) \quad \text{uniformly on } [0,1], \text{ and}$$

$$\lim_{\epsilon \to 0+} y'(t,\epsilon) = \begin{cases} Y_L'(t) & 0 \le t < t_0 \\ Y_R'(t) & t_0 < t \le 1 \end{cases}$$

uniformly on $[0,t_0 - \delta]$ and on $[t_0 + \delta, 1]$ for any $\delta > 0$. Furthermore, for ϵ small enough,

$$\mu_1 < y'(t_0,\epsilon) < \mu_2 .$$

The solution is unique in the sense that there is no other solution of (5.1) which lies in a sufficiently small neighborhood of $Y(t)$ throughout $[0,1]$ for small $\epsilon > 0$.

Using our basic estimates we can approach (5.1) from the theory of second order equations and shed some light on the conditions imposed in this theorem. We make the following observation.

<u>Lemma 5.1</u> Suppose there is an $\epsilon_0 > 0$ such that there exists a solution $y(t,\epsilon)$ of (5.1) for every ϵ with $0 < \epsilon \le \epsilon_0$. Suppose $[a,b] \subset [0,1]$ is an interval on which

(5.4) $$|y(t,\epsilon)| + |y'(t,\epsilon)| \le M$$

and

(5.5) $$|f_{y'}(t,y(t,\epsilon), y'(t,\epsilon),\epsilon)| \ge k > 0$$

for two positive constants M and k. Then, there is a function $Y(t) \in C^1(a,b)$ which satisfies

$$f(t,Y(t),Y'(t),0) = 0 \tag{5.6}$$

and there is a sequence $\epsilon_n \to 0+$ such that

$$\begin{cases} \lim\limits_{\epsilon_n \to 0+} y(t,\epsilon_n) = Y(t) \quad \text{uniformly on } [a,b], \text{ and} \\ \lim\limits_{\epsilon_n \to 0+} y'(t,\epsilon_n) = Y'(t) \quad \text{uniformly on } [a+\delta,b-\delta]\,. \end{cases} \tag{5.7}$$

Proof: Let $u(t,\epsilon) = y'(t,\epsilon)$. Then

$$\epsilon u'' - f_{y'}(t,y(t,\epsilon),y'(t,\epsilon),\epsilon)u' = B(t) \qquad (a \le t \le b)$$

where $B(t)$ is a bounded function. Using Theorem 2.7 we find that $u'(t,\epsilon) = y''(t,\epsilon)$ is uniformly bounded on every subinterval $[a+\delta,b-\delta]$, and, the lemma follows from the Ascoli-Arzela theorem [8].

While it is difficult to go much farther in general, we are able to treat a large class of problems which are quadratic in y'. First, we consider some particular examples. The problem (see [14],[29])

$$\begin{cases} \epsilon y'' + (y')^2 = 1 \qquad (0 \le t \le 1) \\ y(0) = A, \quad y(1) = B \end{cases}$$

with

$$0 < B - A < 1$$

is easily solved. The limit function is the angular solution

$$Y(t) = \begin{cases} A - t & 0 \le t \le \dfrac{1 - (B - A)}{2} \\ B + (t-1) & \dfrac{1 - (B - A)}{2} \le t \le 1. \end{cases}$$

On the other hand, if

$$1 \le B - A,$$

then

$$Y(t) = B + (t - 1) \qquad (0 < t \le 1) .$$

Another interesting example is

$$(5.8) \qquad \begin{cases} \epsilon y'' + (y')^2 - |y'|y = 0 & (0 \le t \le 1) \\ y(0) = A < B = y(1). \end{cases}$$

For this problem we have three cases.

<u>Case 1:</u> If $B \ge 0$,

$$(5.8a) \qquad Y(t) = B, \qquad (0 < t \le 1) .$$

<u>Case 2:</u> If $B < 0$ and $Be < A$,

$$(5.8b)\qquad Y(t) = \begin{cases} A & 0 \le t \le \ln \frac{Be}{A}, \\ Be^{1-t} & \ln \frac{Be}{A} \le t \le 1. \end{cases}$$

Case 3: If $B < 0$ and $Be \geq A$,

$$(5.8c)\qquad Y(t) = Be^{1-t} \qquad (0 < t \leq 1).$$

Consider now the general boundary value problem

$$(5.9)\qquad \begin{cases} \epsilon y'' + p(t,y)(y')^2 + g_0(t,y)y' + g_1(t,y)|y'| - \beta(t,y)y = f(t,y) \\ \qquad\qquad\qquad\qquad (0 \leq t \leq 1) \\ y(0) = A < y(1) = B, \quad B > 0 . \end{cases}$$

The coefficients $p(t,y)$, $g_0(t,y)$, $g_1(t,y)$, $\beta(t,y)$, $f(t,y)$ are assumed to be continuously differentiable with respect to t and y for $0 \leq t \leq 1$ and $|y| < \infty$. As usual, we assume that

$$(5.9a)\qquad \beta(t,y) \geq 0.$$

Moreover, we assume that there are constants $p_0 > 0$ and $F_0 < \infty$ such that

$$(5.9b)\qquad \begin{cases} 0 < p_0 \le p(t,y) & (0 \le t \le 1,\ |y| < \infty) \\ |f(t,y)| \le F_0 & (0 \le t \le 1,\ |y| < \infty). \end{cases}$$

We also assume that

$$(5.9c)\qquad f(t,y) \geq 0 \text{ if } y \geq 0.$$

Clearly, if there is a y_0 such that

(*) $$f(t,y) + \beta(t,y)y \le 0 \quad \text{for} \quad y \le y_0 ,$$

then the basic results of section 2 imply the existence of an a-priori lower bound for any solution $y(t,\epsilon)$. In that case, we could assume that the coefficients $g_0(t,y)$, $g_1(t,y)$ are bounded for all negative y. However, instead of the strong assumption (*), we assume that there is a constant g such that

(5.9d) $$g_0(t,y) - g_1(t,y) \le g \quad \text{for} \quad y \le B.$$

Having described our basic problem we turn to the question of a-priori estimates for $y(t,\epsilon)$ and $y'(t,\epsilon)$.

Lemma 5.2 Let $y(t,\epsilon)$ be a solution of (5.9) and let (5.9a - 5.9d) hold. Then

(5.10) $$y(t,\epsilon) \le B \quad \text{for} \quad 0 \le t \le 1, \text{ and } \quad y'(1,\epsilon) \ge 0.$$

Proof: Apply the maximum principle.

Lemma 5.3 Let $y(t,\epsilon)$ be a solution of (5.9) as above. Let

(5.11a) $$\beta_0 = \max \{\beta(t,y) \mid 0 \le t \le 1, \ 0 \le y \le B\}$$

and

(5.11b) $$M_0 = (2p_0)^{-1} \{g + [g^2 + 4 (F_0 + \beta_0 B) p_0]^{\frac{1}{2}} \} .$$

Then

(5.12a) $$y'(t,\epsilon) \geq -M_0,$$

and hence

$$y(t,\epsilon) \geq (A-M_0).$$

Proof: Since (5.10) holds, if $y'(t,\epsilon)$ assumes a negative minimum at a point $t_0 \in [0,1]$, we have

$$y''(t_0,\epsilon) \geq 0.$$

Hence

$$p_0[y'(t_0,\epsilon)]^2 \leq (F_0 + \beta_0 B) + g|y'(t_0,\epsilon)|$$

and the lemma follows at once.

Lemma 5.4 Let $y(t,\epsilon)$ be a solution of (5.9) as above. Let

$$h = \max \{-[g_0(t,y) + g_1(t,y]\,|\, 0 \leq t \leq 1,\ A-M_0 \leq y \leq B\}$$

and

$$M_1 = (2p_0)^{-1}\{h + [h^2 + 4(F_0 + \beta B)p_0]^{\frac{1}{2}}\}.$$

Let $t_0 \in [0,1)$ and suppose

(5.13a) $$y'(t_0,\epsilon) \leq 0.$$

Then

(5.13b) $$y'(t,\epsilon) \leq M_1 \qquad (t_0 \leq t \leq 1).$$

Proof: If $y'(t,\epsilon)$ assumes a positive maximum at a point $t_1 \in [t_0,1]$, we have $y''(t_1,\epsilon) \geq 0$ and

$$p_0(y'(t_1,\epsilon))^2 \leq h\, y'(t_1,\epsilon) + (F + \beta_0 B),$$

which proves the lemma.

Lemma 5.5 Let $y(t,\epsilon)$ be a solution of (5.9) as above. Suppose there is an interval $[0,b]$ such that

$$(5.14) \qquad y'(t,\epsilon) \geq 0 \qquad (0 \leq t \leq b)\,.$$

Let

$$M_2 = (B-A) + (F_0 + \beta_0 B).$$

Then

$$(5.16) \qquad 0 \leq y'(t,\epsilon) \leq \frac{M_2}{t} + M_1 + \frac{(1+h)}{p_0} \qquad (0 < t \leq b)\,.$$

Proof: Let $t_1 \in [0,b]$ be the point at which $y'(t,\epsilon)$ assumes its maximum. If $t_1 > 0$ then $y''(t_1,\epsilon) \geq 0$ and (5.13b) holds. So assume that $t_1 = 0$, and let $\alpha \leq b$ be the largest value such that

$$(5.17) \qquad y'(t,\epsilon) \geq (1+h)/p_0 \qquad (0 \leq t \leq \alpha).$$

Of course, if there is no such α, the lemma is true. Applying Theorem 2.7 to $y(t,\epsilon)$ in the interval $(0,\alpha]$ we obtain

$$(5.18a) \qquad 0 \leq y'(t,\epsilon) \leq \frac{M_2}{t} \qquad (0 < t \leq \alpha).$$

If $\alpha = b$, then the lemma is proven. If $\alpha < b$, let $t_2 \in [\alpha,b]$ be the point at which $y'(t,\epsilon)$ assumes its maximum over this smaller interval. If $t_2 > \alpha$, we argue as above and

(5.18b) $$y'(t,\epsilon) \le M_1 \qquad (\alpha \le t \le b).$$

On the other hand, if $\alpha = t_2$, then

(5.18c) $$y'(t,\epsilon) \le y'(\alpha,\epsilon) = (1+h)/p_0 \qquad (\alpha \le t \le b).$$

In either case, combining (5.18a), (5.18b) and (5.18c) proves the lemma.

Collecting these results, we have the following basic set of estimates which are independent of ϵ.

Theorem 5.1 Let $y(t,\epsilon)$ be a solution of (5.9). Let (5.9a-5.9d) hold. Then

(5.19a) $$A-M_0 \le y(t,\epsilon) \le B.$$

If $y'(t,\epsilon)$ assumes its positive maximum at $t = 0$, then

(5.19b) $$|y'(t,\epsilon)| \le \frac{M_2}{t} + M_0 + M_1 + \frac{(1+h)}{p_0} \qquad (0 < t \le b).$$

On the other hand, if $y'(t,\epsilon)$ does not assume its positive maximum at $t=0$, then

(5.19c) $$|y'(t,\epsilon)| \le M_0 + M_1.$$

Theorem 5.2 There is a solution $y(t,\epsilon)$ of equation (5.3a) subject to the conditions (5.9a - 5.9d).

<u>Proof</u>: Let $y(t,\epsilon)$ be a solution. Let $W(s)$ be the bounded discontinuous function

$$W(s) = p(s,y(s,\epsilon))y'(s,\epsilon) + g_0(s,y(s,\epsilon)) + g_1(s,y(s,\epsilon)) \operatorname{sgn} y'(s,\epsilon),$$

and let $Q(t)$ be the continuous function

$$Q(t) = \int_0^t W(s)ds.$$

Using the estimates on $y(t,\epsilon)$ we can bound $Q(t)$ from below and above. Then, $y(t,\epsilon)$ is a solution of

$$\left(e^{\frac{Q(t)}{\epsilon}} y'\right)' = \frac{e^{\frac{Q(t)}{\epsilon}} \Omega(t)}{\epsilon}$$

where

$$\Omega(t) = f(t,y(t,\epsilon)) + \beta(t,y(t,\epsilon))y(t,\epsilon).$$

Using (2.11) we are able to obtain a bound on $y'(0,\epsilon)$. This bound and Theorem 5.1 allow us to bound $y'(t,\epsilon)$, $0 \le t \le 1$. Then we can apply the Schauder fixed point theorem to show the existence of a solution.

Having proved the existence of solutions for $\epsilon > 0$ and having obtained some basic estimates (Theorem 5.1) which are independent of ϵ, we are in a position to discuss the asymptotic behavior of $y(t,\epsilon)$ as $\epsilon \to 0+$. We collect several basic convergence results.

Lemma 5.6. For every $\epsilon > 0$ let $y(t,\epsilon)$ be a solution of (5.9). Then there is a sequence $\epsilon_n \to 0+$ and a function $Y(t) \in C[0,1]$ such that for every $\delta \in (0,1)$

$$\lim_{\epsilon_n \to 0+} \ \max_{\delta \le t \le 1} |y(t,\epsilon_n) - Y(t)| \ . \tag{5.20a}$$

Moreover, the function $Y(t)$ is absolutely continuous and satisfies a Lipschitz condition on every interval $[\delta,1]$. Finally,

$$\lim_{\epsilon_n \to 0+} y'(t,\epsilon_n) = Y'(t) \ \text{ weakly in } L^2(\delta,1). \tag{5.20b}$$

Proof: Apply the Ascoli-Arzela theorem and the weak compactness of bounded sequences.

Corollary: Suppose $(a,b) \subset (0,1)$ is an interval on which, for all $n \ge n_0$,

$$H(t,y(t,\epsilon_n)) \equiv g_0(t,y(t,\epsilon_n)) + g_1(t,y(t,\epsilon_n)) \cdot \operatorname{sgn} y'(t,\epsilon_n) \tag{5.21a}$$

is continuous and

$$2p(t,y(t,\epsilon_n)) \cdot y'(t,\epsilon_n) + H(t,y(t,\epsilon_n)) \le -k_1 < 0 \tag{5.21b}$$

for some constant k_1. Then, for every $\delta > 0$,

$$\lim_{\epsilon_n \to 0+} \ \max_{a \le t \le b-\delta} |y'(t,\epsilon_n) - Y'(t)| = 0. \tag{5.22}$$

Similarly, if $(a,b) \subset (0,1)$ is an interval on which (5.21a) holds and, for all $n \ge n_0$,

$$2p(t,y(t,\epsilon_n))y'(t,\epsilon_n) + H(t,y(t,\epsilon_n)) \ge k_2 > 0 \ , \tag{5.23}$$

then

$$(5.24)\qquad \lim_{\epsilon_n \to 0+}\ \max_{a+\delta \le t \le b} |y'(t,\epsilon_n) - Y'(t)| = 0.$$

Proof. As in the proof of Lemma 5.1, the application of Theorem 2.7 to the equation fo $y'(t,\epsilon)$ yields (5.22). Because of the uniqueness of the limit function $Y'(t)$ we can dispense with the subsequence.

Unfortunately, (5.21b) is a rather stringent condition and this result is not quite strong enough. Our next two lemmas give us "strong" convergence of $y'(t,\epsilon_n)$.

Lemma 5.7 Let $\epsilon_n \to 0+$ and let $\{y(t,\epsilon_n)\}$ be a convergent sequence as in Lemma 5.6. Let $Y(t)$ be the limit function. Let $(a,b) \subset (0,1)$ be an interval in which (5.21a) holds and on which $y'(t,\epsilon_n)$ is bounded. Then

$$(5.25)\qquad \lim_{\epsilon_n \to 0+} \int_a^b [y'(t,\epsilon_n)]^2 dt = \int_a^b \frac{f(t,Y(t)) + \beta(t,Y(t))Y(t) - H(t,Y(t))Y'(t)}{p(t,Y(t))}\, dt.$$

Proof: Divide equation (5.9) by $p(t,y(t,\epsilon_n))$ and integrate over the interval (a,b). The uniform convergence of $H(t,y(t,\epsilon_n))/p(t,y(t,\epsilon_n))$ together with the weak convergence of $y'(t,\epsilon_n)$ implies the convergence of the integral on the right hand side of equation (5.25). An easy integration by parts, together with the boundedness of $y'(t,\epsilon_n)$, disposes of the term

$$\epsilon_n \int_a^b \frac{y''(t,\epsilon_n)}{p(t,y(t,\epsilon_n))}\, dt.$$

Lemma 5.8 Under the hypotheses of Lemma 5.7 we have:

(5.26a) $$\lim_{\epsilon_n \to 0+} \int_a^b |y'(t,\epsilon_n)|^2 dt = \int_a^b |Y'(t)|^2 dt,$$

(5.26b) $$\lim_{\epsilon_n \to 0+} y'(t,\epsilon_n) = Y'(t) \text{ in } L^2(a,b)$$

(5.26c) $$p(t,Y(t))(Y'(t))^2 = f(t,Y(t)) + \beta(t,Y(t))Y(t) - H(t,Y(t))Y'(t).$$

Proof: Let

$$z(t,\epsilon_n) = (y'(t,\epsilon_n))^2 \qquad (a \leq t \leq b).$$

On the interval (a,b) these functions are uniformly bounded. Hence there is a subsequence $\epsilon_{n'}$ so that the function $z(t,\epsilon_{n'})$ converge weakly (in $L^2(a,b)$) to a function $Z(t)$. The Cesáro means of this subsequence converge strongly ([26, p. 80]), and a subsequence of the Cesáro means converges almost everywhere to $Z(t)$. Since the same statements apply to $y'(t,\epsilon_n)$, and this sequence of functions converges weakly to $Y'(t)$, we see that

$$Z(t) = (Y'(t))^2.$$

Once more we divide (5.9) by $p(t,y(t,\epsilon_n))$. Evaluate the Cesáro means and integrate over the interval (a,b). Since the right hand side converges we see that (5.26a) holds. However, (5.26a) and the weak convergence implies (5.26b) and (5.26c).

At this point we would like to find general conditions which enable us to find intervals (a,b) on which $H(t,y(t,\epsilon_n))$ is continuous and for which we are able to determine the appropriate root of (5.26c). Of course, we should also like to find conditions which allow us to apply the corollary to Lemma 5.6. In any particular example one can probably carry through a large part of this analysis. However, it is not apparent how to do this in great generality, although there are some useful results. Let

$$
(5.27) \qquad \begin{cases} E(t,y) = [f(t,y) + \beta(t,y)y], \\ S(t,y) = [g_0(t,y) + g_1(t,y)], \\ D(t,y) = [g_0(t,y) - g_1(t,y)]. \end{cases}
$$

Let $u(t)$ be the unique solution of

$$
(5.28) \qquad \begin{cases} u' = [2p(t,u)]^{-1} \{-S(t,u) + [S^2(t,u) + 4p(t,u)E(t,u)]^{\frac{1}{2}}\} \\ u(1) = B \end{cases}
$$

which exists in some interval $(a,1]$. We shall assume that there are values $a_0 \in (a,1)$ and $y_0 \leq 0$ such that

$$
(5.28a) \qquad E(t,y) \geq 0 \qquad (a_0 < t \leq 1, \ y_0 \leq y \leq B)
$$

and

$$
(5.28b) \qquad \text{If } E(t,y) = 0, \text{ then } S(t,y) \geq 0 \qquad (a_0 \leq t \leq 1, \ y_0 \leq y \leq B).
$$

Similarly, let $v(t)$ be the unique solution of

$$
(5.29) \qquad \begin{cases} v' = 2[p(t,v)]^{-1} \{-D(t,v) - [D^2(t,v) + 4p(t,v)E(t,v)]^{\frac{1}{2}}\} \\ v(0) = A \end{cases}
$$

which exists in some interval $[0,b)$. Sometimes we will assume that there are values $b_o \in (0,b)$, and $y_0 < A$ such that

$$(5.29a) \qquad E(t,y) \geq 0 \qquad (0 \leq t \leq b_0, \; y_0 \leq y \leq A)$$

and

$$(5.29b) \qquad \text{If} \quad E(t,y) = 0 \text{ then } D(t,y) \leq 0, \qquad (0 \leq t \leq b_0, \; y_0 \leq y \leq A).$$

Lemma 5.9 Let $u(t)$ be the solution of (5.28). Assume that (5.28a) and (5.28b) hold. Let $y(t,\epsilon)$ be a solution of (5.9) subject to the conditions (5.9a-5.9d). Then, there is an interval $[a_1, 1]$ such that

$$(5.30) \qquad \lim_{\epsilon \to 0+} y(t,\epsilon) = u(t) \qquad (a_1 \leq t \leq 1).$$

Proof: For each $\epsilon > 0$ and corresponding $y(t,\epsilon)$, there are values $\alpha(\epsilon)$ and $\gamma(\epsilon)$ such that

$$(5.31a) \quad \begin{cases} y(\alpha(\epsilon),\epsilon) = \max(0,A) & \\ y(t,\epsilon) \geq \max(0,A) & (\alpha(\epsilon) \leq t \leq 1) \\ y(\gamma(\epsilon), \epsilon) \geq 0 & \\ y'(t,\epsilon) \geq 0 & (\gamma(\epsilon) < t \leq 1). \end{cases}$$

Of course,

$$(5.31b) \qquad \gamma(\epsilon) \leq \alpha(\epsilon).$$

Using the estimates of Theorem 5.1, it is easy to obtain an upper bound for $\alpha(\epsilon)$,

$$(5.31c) \qquad \alpha(\epsilon) < \alpha_0 < 1.$$

Let $\{y(t,\epsilon_n)\}$ be a convergent sequence as in Lemma 5.6, and let $Y(t)$ be the limit function. We can also assume that there is a value γ_1 such that

$$\lim_{\epsilon_u \to 0+} \gamma(\epsilon_n) = \gamma_1 \leq \alpha_0.$$

Applying Lemma 5.8 we have, for every $\delta > 0$,

$$Y'(t) \geq 0 \quad \text{and} \quad Y(t) \geq 0 \qquad (\gamma_1 + \delta \leq t \leq 1).$$

On the interval $[\gamma_1 + \delta, 1]$ we have

$$p(t,Y)(Y')^2 = E(t,Y(t)) + S(t,Y(t))Y'(t)$$

and, using (5.28a) and (5.28b), we see that

$$(5.32) \qquad Y(t) = u(t), \qquad (\min(\gamma_1, a_0) \leq t \leq 1).$$

Because the limit function is unique over the interval $\min(\alpha_0, a_0) \leq t \leq 1$, we can dispense with the choice of a sequence $\epsilon_n \to 0+$ and assert that, on this subinterval, the entire family $y(t,\epsilon)$ is convergent to $u(t)$.

Corollary 1 If there is a $\delta > 0$ such that

$$S(t,y) \geq \delta \qquad (a_0 \leq t \leq 1, \quad 0 \leq y \leq B),$$

then for every $\Delta > 0$ we have

$$\lim_{\epsilon \to 0+} \max\{|y'(t,\epsilon) - Y'(t)| \;\big|\; \min(\alpha_0, a_0) + \Lambda \leq t \leq 1\} = 0.$$

Proof: Apply Lemma 5.6.

Corollary 2 Suppose $a_0 = 0$ and

$$(5.33) \qquad u(0) > \max(A, 0).$$

Then

$$(5.34) \qquad \lim_{\epsilon \to 0+} y(t,\epsilon) = u(t), \quad (0 < t \leq 1).$$

Proof: Let $\epsilon_n \to 0+$, $y(t,\epsilon_n)$ and $Y(t)$ be as above. Suppose corollary 2 is false, and let Ω be the smallest value such that

$$Y(t) = u(t) \qquad (\Omega \leq t \leq 1).$$

Considering the definition of $\alpha(\epsilon)$ and $\gamma(\epsilon)$, we have

(5.35) $\quad \gamma_1 = \Omega$ and $Y(\Omega) > 0$.

Thus for ϵ_n sufficiently small, $y(t,\epsilon_n)$ has a positive minimum at $\gamma(\epsilon_n)$. But then

$$y'(t,\epsilon_n) \leq 0 \qquad (0 \leq t \leq \gamma(\epsilon_n)),$$

and hence

$$Y(t) \leq A \qquad (0 \leq t \leq \Omega).$$

If there is a point $t_0 \in (0,\Omega)$ at which $Y(t_0) < A$, then, for ϵ_n small enough, $y(t_0,\epsilon_n) < A$. Since $y'(t,\epsilon_n) \leq 0$ for $t_0 \leq t \leq \gamma(\epsilon_n)$, we cannot have (5.35).

Theorem 5.3. Let $y(t,\epsilon)$ be a solution of (5.9) subject to (5.9a-5.9d). Let $A > 0$, and let $u(t)$ and $v(t)$ be the solutions of (5.28) and (5.29) respectively. Assume that (5.28a), (5.28b), (5.29a), (5.29b) hold with $y_0 = 0$ and $b_0 > a_0$. Let $c_0 \in (a_0,b_0)$ be any point at which

(5.36) $\quad u(c_0) = v(c_0) > 0.$

Then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = \begin{cases} v(t) & 0 \leq t \leq c_0 \\ u(t) & c_0 \leq t \leq 1. \end{cases}$$

Proof: Applying Lemma 5.9 we find a maximal interval $[\Omega, 1]$ in which

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = u(t) \qquad (\Omega \le t \le 1).$$

As usual, let ϵ_n, $y(t,\epsilon_n)$ and $Y(t)$ be as in Lemma 5.6. Let $\alpha(\epsilon_n)$, $\gamma(\epsilon_n)$, γ_1 be as in Lemma 5.9. The complete theorem now follows from an argument similar to the proof of Corollary 2 of Lemma 5.9.

Looking over these latest results we see several classes of problems which are easily treated.

Remark 1: Suppose the coefficients $g_0(t,y)$ $g_1(t,y)$ are bounded for all y. Suppose

$$\beta(t,y) \equiv 0$$

and

$$f(t,y) \ge \delta_0 > 0.$$

Then we may easily discuss the complete problem. We need merely consider the function

$$Z(t,\epsilon) = y(t,\epsilon) + C$$

where C is a sufficiently large constant so that

$$Z(t,\epsilon) \ge 0.$$

Remark 2: Suppose $f(t,y) \equiv 0$ and

$$\beta(t,y) \geq \delta_0 > 0$$

Then $y(t,\epsilon) \geq 0$ if $A \geq 0$. On the other hand, if $A < 0$ and

$$S(t,y)y \leq 0,$$

then

$$y'(t,\epsilon) \geq 0$$

and the limit function $Y(t)$ must be nonnegative. With these insights one can easily discuss the entire problem.

We now turn our attention to equation (5.8). We observe that

$$y'(t,\epsilon) \geq 0. \tag{5.38}$$

If $B > 0$, then (5.8a) follows from Corollary 2 of Lemma 5.9. If $B < 0$, we consider the positive function

$$Z(t,\epsilon) = y(t,\epsilon) - 2A$$

which satisfies

$$\epsilon Z'' = -Z'[Z' + (Z + 2A)]. \tag{5.39}$$

Let $\epsilon_n \to 0+$ and $y(t,\epsilon_n)$ and $Y(t)$ be as in Lemma 5.6. If $u = y'$, then

$$\epsilon u'' + (2u' + y)u' = -u^2 < 0.$$

Thus, $y'(t,\epsilon)$ cannot possess an interior minimum.

We consider two cases.

Case 1: $y'(t,\epsilon_n)$ assumes its maximum at $t = 0$. In this case $y''(t,\epsilon_n) \leq 0$ in the entire interval $0 < t \leq 1$. Hence

$$(5.40) \qquad y'(t,\epsilon_n) + y(t,\epsilon_n) \geq 0$$

and

$$(5.41) \qquad \begin{cases} y'(t,\epsilon_n) \geq -y(t,\epsilon_n) \geq -B \\ 2y'(t,\epsilon_n) + y(t,\epsilon_n) \geq -B, \end{cases}$$

Thus, applying the Corollary to Lemma 5.6 we obtain, for every $\delta > 0$,

$$(5.42) \qquad \lim_{\epsilon_n \to 0+} \max_{\delta \leq t \leq 1} \{ |y(t,\epsilon_n) - Y(t)| + |y'(t,\epsilon_n) - Y'(t)| \} = 0,$$

and

$$(5.43) \qquad Y(t) = Be^{1-t}$$

Since $A \leq Y(t)$, we see that this case can occur only if $A < Be$.

Case 2: $y'(t,\epsilon_n)$ assumes its maximum at some point $\alpha(\epsilon_n) \in (0,1]$. Applying Theorem 5.1, we see that $y(t,\epsilon_n)$ converges uniformly to $Y(t)$. In particular

$$(5.44) \qquad \lim_{t \to 0+} Y(t) = Y(0) = A.$$

Since $Y(t)$ is continuous near $t = 1$ and

(5.45) $\quad [Y'(t)]^2 = Y'Y \qquad$ a. e. on $[0,1]$,

we see that

$$Y(t) \geq Be^{1-t}.$$

Combining this fact with (5.44) we see that

(5.46) $\quad Be \leq A.$

Without loss of generality, we may assume that there is an α_1 such that

(5.47) $\quad \lim_{\epsilon_n \to 0+} \alpha(\epsilon_n) = \alpha_1 .$

Arguing as before we see that

(5.48) $\quad Y(t) = Be^{1-t} \qquad (\alpha_1 \leq t \leq 1),$

and moreover, $y(t,\epsilon_n)$ and $y'(t,\epsilon_n)$ both converge uniformly on the interval $[\alpha_1, 1]$. Furthermore, on the interval $[0,\alpha(\epsilon_n))$ the function $y'(t,\epsilon_n)$ is increasing. Thus, if $t_0 < \alpha_1$ and

(5.48) $\quad \lim_{\epsilon_n \to 0+} y'(t_0,\epsilon_n) = 0,$

we have

$$\lim_{\epsilon_n \to 0+} y'(t,\epsilon_n) = 0 \qquad (0 \leq t \leq t_0).$$

In addition, there will be an $n_0 > 0$ and a constant k such that

$$2y'(t,\epsilon_n) + y(t,\epsilon_n) \leq -k \qquad (n \geq n_0, \quad 0 \leq t \leq t_0).$$

Thus, if (5.48) holds for any subsequence ϵ_n, then

$$Y(t) = A \qquad (0 \leq t \leq t_0).$$

Since (5.45) holds, we see that there is a unique constant t_0 such that

$$(5.50) \qquad Y(t) = \begin{cases} A & 0 \leq t \leq t_0, \\ Be^{1-t} & t_0 \leq t \leq 1. \end{cases}$$

Clearly, t_0 is determined by the requirement that $Y(t) \in C[0,1]$. In addition, $y'(t,\epsilon_n)$ converges uniformly to $Y'(t)$ on every interval $[0, t_0 - \delta]$ and every interval $[t_0 + \delta, 1]$.

This completes the discussion of equation (5.8). The interested reader will see many more ways in which the estimates of this section can be used in particular examples.

6. Differential Inequalities

There is an extensive literature on the use of differential inequalities and comparison theorems for estimating solutions of differential equations, and the maximum principle can be regarded as a special case of these general results. It warrants special treatment because it is especially simple and powerful. In this section, we treat a variety of examples that are based on more general comparison theorems than the maximum principle.

The basic requirements for our exposition are: (1) a fundamental family of comparison problems, whose solutions are easily analyzed, (2) a class of differential equations which some member of the comparison family satisfies as an inequality, and (3) a theorem relating the solutions of the differential equation and inequality. Bailey, Shampine, and Waltman [1] treat the class of nonlinear differential equations

$$
(6.1) \qquad \begin{cases} \epsilon y''(t) + f(t, y(t), y'(t), \epsilon) = 0 & (a < t < b) \\ y(a) = A, \quad y(b) = B, \end{cases}
$$

where $f(t, y, y', \epsilon)$ is a continuous function satisfying the one-sided Lipschitz conditions

$$
(6.2)\quad \begin{cases} K_1(\epsilon)(y_1 - y_2) \le f(t,y_1,y',\epsilon) - f(t,y_2,y',\epsilon) \le K_2(\epsilon)(y_1 - y_2) \\ \qquad\qquad\qquad\qquad\qquad \text{if } y_1 \ge y_2, \\ L_1(\epsilon)(y_1' - y_2') \le f(t,y,y_1',\epsilon) - f(t,y,y_2',\epsilon) \le L_2(\epsilon)(y_1' - y_2') \\ \qquad\qquad\qquad\qquad\qquad \text{if } y_1' \ge y_2'. \end{cases}
$$

In order to decide when (6.1) has a unique solution, we define a function $\alpha(L,K)$ by

$$
\alpha(L,K) = \begin{cases} \dfrac{2}{(4K - L^2)^{1/2}} \cos^{-1} \dfrac{L}{2\sqrt{K}} & \text{if } 4K - L^2 > 0 \\ \dfrac{2}{(L^2 - 4K)^{1/2}} \cosh^{-1} \dfrac{L}{2\sqrt{K}} & \text{if } 4K - L^2 < 0,\ L > 0,\ K > 0 \\ \dfrac{2}{L} & \text{if } 4K - L^2 = 0,\ L > 0 \\ +\infty & \text{otherwise}, \end{cases}
$$

and we let $\beta(L,K) = \alpha(-L,K)$. Then if there exists an $\epsilon_0 > 0$ such that

$$
(6.3)\qquad 0 < b - a < \alpha\left(\frac{L_2(\epsilon)}{\epsilon}, \frac{K_2(\epsilon)}{\epsilon}\right) + \beta\left(\frac{L_1(\epsilon)}{\epsilon}, \frac{K_2(\epsilon)}{\epsilon}\right)
$$

for $0 < \epsilon \le \epsilon_0$, (6.1) has a unique solution for $0 < \epsilon \le \epsilon_0$ [1 , p. 96]. We remark that the following are convenient sufficient conditions which guarantee that (6.3) is satisfied for ϵ small enough:

(i) $K_2(\epsilon) < 0$ if $0 < \epsilon \le \epsilon_0$,

(ii) $\lim_{\epsilon \to 0+} K_2(\epsilon) = \bar{K}_2 \ge 0$ and

$\lim_{\epsilon \to 0+} L_2(\epsilon) = \bar{L}_2 < 0$,

(iii) $\lim_{\epsilon \to 0+} K_2(\epsilon) = \bar{K}_2 \ge 0$ and

$\lim_{\epsilon \to 0+} L_1(\epsilon) = \bar{L}_1 > 0$.

In order to generate the family of comparison problems, we observe that the inequalities in (6.2) can be written as

$$G_1(y_1-y_2, y_1'-y_2', \epsilon) \le f(t, y_1, y_1', \epsilon) - f(t, y_2, y_2', \epsilon) \le G_2(y_1-y_2, y_1'-y_2', \epsilon),$$

where

$$G_1(y, y', \epsilon) = \begin{cases} K_1(\epsilon)y + L_1(\epsilon)y' & \text{if } y \ge 0,\ y' \ge 0 \\ K_1(\epsilon)y + L_2(\epsilon)y' & \text{if } y \ge 0,\ y' \le 0 \\ K_2(\epsilon)y + L_1(\epsilon)y' & \text{if } y \le 0,\ y' \ge 0 \\ K_2(\epsilon)y + L_2(\epsilon)y' & \text{if } y \le 0,\ y' \le 0 \end{cases}$$

and

$$G_2(y, y', \epsilon) = \begin{cases} K_2(\epsilon)y + L_2(\epsilon)y' & \text{if } y \ge 0,\ y' \ge 0 \\ K_2(\epsilon)y + L_1(\epsilon)y' & \text{if } y \ge 0,\ y' \le 0 \\ K_1(\epsilon)y + L_2(\epsilon)y' & \text{if } y \le 0,\ y' \ge 0 \\ K_1(\epsilon)y + L_1(\epsilon)y' & \text{if } y \le 0,\ y' \le 0 . \end{cases}$$

For $i = 1$ and 2 the differential equations

$$(6.4)\quad \begin{cases} \epsilon u_i''(t) + G_i(u_i(t), u_i'(t), \epsilon) + f(t,0,0,\epsilon) = 0 & (a < t < b) \\ u_i(a) = A,\ u_i(b) = B \end{cases}$$

are included in the family defined by (6.1). Thus if (6.3) is satisfied, unique solutions $u_i(t,\epsilon)$ exist to (6.4). Furthermore, a basic comparison result [1 , p. 96] states that in this case we have

$$(6.5)\quad u_1(t,\epsilon) \le y(t,\epsilon) \le u_2(t,\epsilon) \qquad (a \le t \le b).$$

The comparison equations (6.4) are relatively simple, for they are linear equations with constant coefficients in regions where $u_i(t)$ and $u_i'(t)$ have one sign. Indeed, if $f(t,0,0,\epsilon) \equiv 0$, the exact solutions to (6.4) can be explicitly computed [2]. We now consider some examples for which $\lim_{\epsilon \to 0+} u_1(t,\epsilon) = \lim_{\epsilon \to 0+} u_2(t,\epsilon)$. We see from (6.5) that we then also have $\lim_{\epsilon \to 0+} y(t,\epsilon) = \lim_{\epsilon \to 0+} u_1(t,\epsilon)$.

<u>Theorem 6.1</u>. Suppose that for $0 < \epsilon \le \epsilon_0$ the problem

$$\begin{cases} \epsilon y''(t) + g(t, y(t), y'(t), \epsilon) y'(t) = 0 & (a < t < b) \\ y(a) = A,\ y(b) = B \end{cases}$$

has at least one solution $y(t,\epsilon)$ such that $G_1 \le g(t, y(t), y'(t), \epsilon) \le G_2$ with $G_1 G_2 > 0$. Then, for $a < t < b$,

$$(6.6)\quad \lim_{\epsilon \to 0+} y(t,\epsilon) = \begin{cases} A & \text{if } G_1 < 0 \\ B & \text{if } G_1 > 0 . \end{cases}$$

Proof: Let $w(t,\epsilon)$ be the solution to

$$\begin{cases} \epsilon w''(t) + Gw'(t) + Hw(t) = 0 & (a < t < b) \\ w(a) = A, \ w(b) = B, & \end{cases}$$

with $H < 0$. Then, for $a < t < b$,

$$\lim_{\epsilon \to 0+} w(t,\epsilon) = \begin{cases} A \exp\left[-\frac{H}{G}(t-a)\right] & \text{if } G < 0,\ H \le 0 \\ B \exp\left[-\frac{H}{G}(t-b)\right] & \text{if } G > 0,\ H \le 0 \\ 0 & \text{if } G = 0,\ H < 0\,. \end{cases}$$

If $H \le 0$, it follows from the maximum principle that $w'(t,\epsilon)$ has one sign. Thus the bounding functions $u_i(t,\epsilon)$ in (6.5) have a common limit, which proves the theorem.

The statement in (6.6) is typical of those which follow in this section. However, it should be remembered that we actually prove a much stronger result. That is, we exhibit computable functions $u_i(t,\epsilon)$ such that $u_1(t,\epsilon) \le y(t,\epsilon) \le u_2(t,\epsilon)$, and for these cases the $\mathring{u}_i(t,\epsilon)$ collapse to a common limit function as $\epsilon \to 0+$. For many combinations of boundary conditions and Lipschitz constants the bounding functions $u_i(t,\epsilon)$ do not have a common limit, and the theorems we state are a sample of the successful cases.

The method of proof of Theorem 6.1 also yields the following two results.

Theorem 6.2. Suppose that $0 \leq A \leq B$, and for $0 < \epsilon \leq \epsilon_0$ the problem

$$\begin{cases} \epsilon y''(t) + h(t,y(t),y'(t),\epsilon)y(t) = 0 & (a < t < b) \\ y(a) = A, \quad y(b) = B \end{cases}$$

has at least one solution $y(t,\epsilon)$ such that $h(t,y(t),y'(t),\epsilon) \leq H_1 < 0$. Then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = 0 \qquad (a < t < b).$$

Theorem 6.3. Suppose that for $0 < \epsilon \leq \epsilon_0$ the problem

$$\begin{cases} \epsilon y''(t) + g(t,y(t),y'(t),\epsilon)y'(t) + h(t,y(t),y'(t),\epsilon)y(t) = 0 & (a < t < b) \\ y(a) = A, \quad y(b) = B \end{cases}$$

has at least one solution $y(t,\epsilon)$ such that $h(t,y(t),y'(t),\epsilon) \leq 0$. Also assume either that $A = 0$ and $g(t,y(t),y'(t),\epsilon) \leq G_1 < 0$ or that $B = 0$ and $g(t,y(t),y'(t),\epsilon) \geq G_2 > 0$. Then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = 0 \qquad (a < t < b).$$

The last theorem can easily be extended to apply to singular perturbation problems with two small parameters (cf. [19 , sec. 4] and the references given there).

Theorem 6.4. Suppose that for $0 < \epsilon \leq \epsilon_0$ and $0 < \mu \leq \mu_0$ the problem

$$\begin{cases} \epsilon y'' + \mu g(t,y,y',\epsilon,\mu)y' + h(t,y,y',\epsilon,\mu)y = 0 & (a < t < b) \\ y(a) = 0, y(b) = B \geq 0. \end{cases}$$

has at least one solution $y(t,\epsilon,\mu)$ such that

$G_1 \leq g(t,y(t),y'(t),\epsilon,\mu) < G_2$ and $H_1 \leq h(t,y(t),y'(t),\epsilon,\mu) < H_2 < 0$.

If we assume that either $\frac{\epsilon}{\mu^2} \to 0+$ or $\frac{\mu^2}{\epsilon} \to 0+$, then

$$\lim_{\substack{\epsilon \to 0+ \\ \mu \to 0+}} y(t,\epsilon,\mu) = 0 \qquad (a < t < b).$$

<u>Proof</u>. Using the usual comparison functions, we consider the problem

$$\begin{cases} \epsilon w''(t) + \mu G w'(t) + H w(t) = 0 & (a < t < b) \\ w(a) = 0, \quad w(b) = B, \end{cases}$$

where $H < 0$. With the assumptions made in the theorem, it is easy to show that

$$\lim_{\substack{\epsilon \to 0+ \\ \mu \to 0+}} w(t,\epsilon,\mu) = 0 \qquad (a < t < b),$$

and the theorem follows easily from this result.

It should be noted that Theorem 6.4 can be applied to turning point problems, since we have made no restrictions on the sign of $g(t,y,y',\epsilon,\mu)$. Asymptotic expansions for the case when $g \equiv g(t)$ and $h \equiv h(t)$ with $|g(t)| \geq G_1 > 0$ are given by

O'Malley in [18 , Sec. 4] and [19 , Sec. 4.B].

We now turn to a different type of application of these comparison results. Consider the quasilinear problem in (2.3) with $\gamma(t,y,\epsilon) \equiv 0$. In the preceding sections we have required $\beta(t,y,\epsilon) \geq 0$ so that the maximum principle can be used. With the use of these general differential inequalities, we can relax this restriction. As an example of the application of this technique, we will prove an analog of Theorem 2.4 without the sign restriction on $\beta(t,y,\epsilon)$. We first state a preliminary result.

Lemma 6.1. For $i = 1,2$ consider the comparison problems

$$\begin{cases} \epsilon u_i''(t) + G_i(u_i(t), u_i'(t), \epsilon) = 0 & (a < t < b) \\ u_i(a) = A(\epsilon),\ u_i(b) = 0. \end{cases}$$

Assume that $L_i > 0$ and $K_1 = k_{11} + \frac{k_{12}}{\epsilon}$ with $k_{12} \leq 0$. Then there is an $\epsilon_0 > 0$ such that

$$|u_i(t,\epsilon)| \leq |A(\epsilon)| \exp\left[-\frac{L_1}{2\epsilon}(t-a)\right] \qquad (a \leq t \leq b)$$

for $0 < \epsilon < \epsilon_0$ and $i = 1,2$.

Proof. Let $\Delta(L,K) = \frac{1}{\epsilon}K - \frac{1}{4\epsilon^2}L^2$. With our assumptions, there exists an $\epsilon_0 > 0$ such that $\Delta(L_i, K_j) < 0$ for $0 < \epsilon \leq \epsilon_0$, $i = 1,2$, and $j = 1,2$. If we let $k(L,K) = (|\Delta(L,K)|)^{\frac{1}{2}}$, it is easy to show that

$$u_i(t,\epsilon) = A(\epsilon)\ \frac{\sinh[k(L_{3-i},K_i)(t-b)]}{\sinh[k(L_{3-i},K_i)(a-b)]}\ \exp\left[-\frac{L_{3-i}}{2\epsilon}(t-a)\right],$$

and the lemma follows easily from this result.

<u>Theorem 6.5</u>. Suppose that for $0 < \epsilon \le \epsilon_0$ the problem

$$\begin{cases} \epsilon y'' + \alpha(t,y,y',\epsilon)y' - \beta(t,y,y',\epsilon)y = 0 & (a < t < b) \\ y(a) = A(\epsilon),\ y(b) = 0 \end{cases}$$

has at least one solution $y(t,\epsilon)$ such that

$0 < \alpha_0 \le \alpha(t,y(t),y'(t),\epsilon) \le \alpha_1$ and $|\beta(t,y(t),y'(t),\epsilon)| \le \beta_0$. Then for $0 < \epsilon \le \epsilon_1$,

$$|y(t,\epsilon)| \le |A(\epsilon)| \exp\left[-\frac{\alpha_0}{2\epsilon}(t-a)\right] \qquad (a \le t \le b).$$

<u>Proof</u>. Define a comparison function $\phi(t)$ by

$$\phi(t) = y(t,\epsilon)\exp\left[\frac{\alpha_0}{3\epsilon}\left(\frac{t}{3} - a\right)\right].$$

Then $\phi(t)$ satisfies the equation

$$\begin{cases} \epsilon\phi'' + \left(\alpha - \frac{2\alpha_0}{3}\right)\phi' + \left(\frac{\alpha_0}{3\epsilon}\left(\frac{\alpha_0}{3} - \alpha\right) - \beta\right)\phi = 0 & (a \le t \le b). \\ \phi(a) = A(\epsilon),\ \phi(b) = 0, \end{cases}$$

and the proof now follows easily from Lemma 6.1.

If we expand our class of comparison problems, we can treat certain kinds of turning point problems. For some similar recent work in this area we refer to O'Malley [22] and Dorr [10].

Consider the problem

$$\begin{cases} \epsilon u''(t) + t^k G(t)u'(t) = 0 & (a < t < b) \\ u(a) = A,\ u(b) = B\ , & \end{cases}$$

where k is a nonnegative integer, $a < 0 < b$, and

$$G(t) = \begin{cases} G_1 & \text{if } a \le t \le 0 \\ G_2 & \text{if } 0 < t \le b \end{cases}$$

with $G_1 G_2 > 0$. If $k = 0$ we require $G_1 = G_2$, so that the function $t^k G(t)$ is continuous for all values of k. We can find the asymptotic behavior of $u(t,\epsilon)$ by examining the following cases:

(i) $G(t) < 0$, k even.

$$\lim_{\epsilon \to 0+} u(t,\epsilon) = A \qquad (a \le t < b)$$

(ii) $G(t) < 0$, k odd.

$$\lim_{\epsilon \to 0+} u(t,\epsilon) = \begin{cases} A & \text{if } \dfrac{G_1}{G_2} < (\dfrac{b}{a})^{k+1} \\ [(\dfrac{b}{b-a})A + (\dfrac{-a}{b-a})B] & \text{if } \dfrac{G_1}{G_2} = (\dfrac{b}{a})^{k+1} \\ B & \text{if } \dfrac{G_1}{G_2} > (\dfrac{b}{a})^{k+1} \end{cases}$$

(iii) $G(t) > 0$, k even.

$$\lim_{\epsilon \to 0+} u(t,\epsilon) = B \qquad (a < t \le b)$$

(iv) $G(t) > 0$, k odd.

$$\lim_{\epsilon \to 0+} u(t,\epsilon) = \begin{cases} A & \text{if} \quad a \le t < 0 \\ B & \text{if} \quad 0 < t \le b . \end{cases}$$

With the use of the appropriate comparison problems and a differential inequality result such as Theorem 1.10 in [25 , p. 18], we then have the following result (cf. [10]).

Theorem 6.6. Assume that k is a nonnegative integer, $a < 0 < b$, and $t^k g(t,y,y',\epsilon)$ is a continuous function. Suppose that for $0 < \epsilon \le \epsilon_0$ the problem

$$\begin{cases} \epsilon y''(t) + t^k g(t,y(t),y'(t),\epsilon)y'(t) = 0 & (a < t < b) \\ y(a) = A,\ y(b) = B \end{cases}$$

has at least one solution $y(t,\epsilon)$ such that $G_1 \le g(t,y(t),y'(t),\epsilon) \le G_2$ with $G_1 G_2 > 0$. Then we have:

(i) If $G_1 < 0$ and k is even,

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = A \qquad (a \le t < b).$$

(ii) If $G_1 > 0$ and k is even,

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = B \qquad (a < t \le b).$$

(iii) If $G_1 > 0$ and k is odd,

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = \begin{cases} A & \text{if} \quad a \le t < 0 \\ B & \text{if} \quad 0 < t \le b. \end{cases}$$

It should be noted that this theorem does not treat the case $G_1 < 0$ and k odd. Indeed, the following result shows that the asymptotic behavior can be somewhat arbitrary for this case.

Theorem 6.7. Let $\alpha \in [0,1]$ be given. Then there exists a piecewise continuous function $g(t)$, which is linear on $[a,0)$ and $[0,b]$ and satisfies $g(t) \leq G_1 < 0$ for $a \leq t \leq b$, such that, if $y(t,\epsilon)$ is the solution to

$$\begin{cases} \epsilon y''(t) + tg(t)y'(t) = 0 & (a < t < b) \\ y(a) = A, \ y(b) = B, \end{cases}$$

then

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = \alpha A + (1-\alpha)B \quad (a < t < b).$$

Proof. See [10].

This type of analysis can also be used to treat problems in which the coefficient of y' has a zero at an end point of the interval.

Theorem 6.8. Assume that k is a nonnegative integer, $b > 0$, and $t^k g(t,y,y',\epsilon)$ is a continuous function. Suppose that for $0 < \epsilon \leq \epsilon_0$ the problem

$$\begin{cases} \epsilon y''(t) + t^k g(t,y(t),y'(t),\epsilon)y'(t) = 0 & (0 < t < b) \\ y(0) = A, \ y(b) = B \end{cases}$$

has at least one solution $y(t,\epsilon)$ such that $G_1 \le g(t,y(t),y'(t),\epsilon) \le G_2$ with $G_1G_2 > 0$. Then, for $0 < t < b$,

$$\lim_{\epsilon \to 0+} y(t,\epsilon) = \begin{cases} A & \text{if } G_1 < 0 \\ B & \text{if } G_1 > 0. \end{cases}$$

The final theorems of this section are of a rather different nature than the preceding results. They are motivated by physical problems arising in the field of chemical engineering. Several authors have considered these problems, and we mention in particular the paper of O'Malley [24] and the further references given there.

Theorem 6.9. Consider the boundary value problem

$$\begin{cases} \epsilon y''(t) - b(t)y'(t) - g(t,y(t)) = 0 & (0 < t < 1) \\ y(0) = A, \ y(1) = B. \end{cases}$$

Assume that $b(0) > 0$, $b'(t) \ge 0$, $A \le B$, $g(t,y)$ is continuous, and $g_y(t,y) \ge 0$. Define $Z(t)$ as the solution of the reduced equation

$$(6.7) \quad \begin{cases} b(t)Z'(t) + g(t,Z(t)) = 0 & (0 < t < 1) \\ Z(0) = A, \end{cases}$$

and $w(t) = w(t,\epsilon)$ as the solution of the boundary value problem

$$\begin{cases} \epsilon w''(t) - b(t)w'(t) - g(t,Z(t)) = 0 & (0 < t < 1) \\ w(0) = A, \ w(1) = B. \end{cases}$$

Assume that $g(t,Z(t)) \ge 0$ and $g_t(t,Z(t)) \le 0$ for $0 \le t \le 1$. Then the functions $y(t,\epsilon)$, $Z(t)$, and $w(t,\epsilon)$ exist for all $\epsilon > 0$ and satisfy

$$z(t) \leq y(t,\epsilon) \leq w(t,\epsilon) \qquad (0 \leq t \leq 1).$$

Furthermore,

$$\lim_{\epsilon \to 0+} w(t,\epsilon) = \lim_{\epsilon \to 0+} y(t,\epsilon) = Z(t) \qquad (0 \leq t < 1).$$

Proof. The existence of the functions follows from previously stated sufficient conditions. We first show that $z(t) \leq y(t)$. Since $Z'(t) < 0$ and $Z(0) = A \leq B$, we see that $Z(1) \leq B$. Since $Z''(t) \geq 0$, we can use the differential inequality Theorem 1.22 in [25 , p. 48] to see that $z(t) \leq y(t)$. The same argument shows that $z(t) \leq w(t)$. Finally, since

$$\epsilon w'' - bw' - g(t,w) = g(t,Z) - g(t,w) \leq 0$$

it follows that $y(t) \leq w(t)$. The convergence of $w(t,\epsilon)$ to $Z(t)$ follows from Theorem 3.1, and this completes the proof of the theorem.

For the physical problems, the boundary conditions are not given by $y(0) = A$ and $y(1) = B$. However, the proof of Theorem 6.9 can be modified to yield the following result.

Theorem 6.10. Consider the boundary value problem

$$\begin{cases} \epsilon y''(t) - b(t)y'(t) - g(t,y(t)) = 0 & (0 < t < 1) \\ y(0) - \epsilon y'(0) = A, \; y'(1) = 0. \end{cases}$$

Assume that $b(0) > 0$, $b'(t) \geq 0$, $g(t,y)$ is continuous, $g(0,A) \geq 0$, and $g_y(t,y) \geq 0$. If we consider the equation

$$(6.8) \qquad r + \frac{\epsilon g(0,r)}{b(0)} = A,$$

then there is a unique solution $r = r_0(\epsilon)$ to eq. (6.8) and $\lim_{\epsilon \to 0+} r_0(\epsilon) = A$. Define $Z(t,\epsilon)$ as the solution of the semi-reduced equation

$$\begin{cases} b(t)\, Z'(t,\epsilon) + g(t,Z(t,\epsilon)) = 0 & (0 < t < 1) \\ Z(0,\epsilon) = r_0(\epsilon), \end{cases}$$

and $w(t) = w(t,\epsilon)$ as the solution of the boundary value problem

$$\begin{cases} \epsilon w''(t) - b(t)\, w'(t) - g(t,Z(t,\epsilon)) = 0 & (0 < t < 1) \\ w(0) - \epsilon w'(0) = A, w'(1) = 0. \end{cases}$$

Assume that $g(t,Z(t,\epsilon)) \geq 0$ and $g_t(t,Z(t,\epsilon)) \leq 0$ for $0 \leq t \leq 1$. Then the functions $y(t,\epsilon)$, $Z(t,\epsilon)$ and $w(t,\epsilon)$ exist for all $\epsilon > 0$ and satisfy

$$z(t,\epsilon) \leq y(t,\epsilon) \leq w(t,\epsilon) \qquad (0 \leq t \leq 1).$$

Furthermore,

$$\lim_{\epsilon \to 0+} w(t,\epsilon) = \lim_{\epsilon \to 0+} y(t,\epsilon) = \lim_{\epsilon \to 0+} Z(t,\epsilon) = Z(t) \qquad (0 \leq t \leq 1),$$

where $Z(t)$ is the solution to the reduced equation (6.7).

Proof. The method of proof is the same as for Theorem 6.9, except for the convergence of $w(t,\epsilon)$ to $Z(t)$. But it follows from Theorem 2.8 that $|y'(t)| \leq M_1$, and the convergence proof can then be carried through in the same way as the proof of Theorem 3.1.

We now give two examples to illustrate the application of the preceding theorems. First, consider the problem

$$\begin{cases} \epsilon y''(t) - y'(t) - \sqrt{y(t)} = 0 & (0 < t < 1) \\ y(0) = 1, \ y(1) = B \geq 1. \end{cases}$$

A simple calculation shows that $Z(t)$ and $w(t,\epsilon)$ of Theorem 6.9 are given by

$$Z(t) = (1 - \frac{1}{2}t)^2,$$

$$w(t,\epsilon) = (1 - \gamma) + (\frac{1}{2}\epsilon - 1)t + \frac{1}{4}t^2 + \gamma \exp(\frac{t}{\epsilon})$$

$$= Z(t) - \gamma + \frac{1}{2}\epsilon t + \gamma \exp(\frac{t}{\epsilon}),$$

where

$$\gamma = (B - \frac{1}{4} - \frac{1}{2}\epsilon)(\exp(\frac{1}{\epsilon}) - 1)^{-1}.$$

Then Theorem 6.9 states that

$$z(t) \leq y(t,\epsilon) \leq w(t,\epsilon) \qquad (0 \leq t \leq 1),$$

and $\lim_{\epsilon \to 0+} w(t,\epsilon) = Z(t)$ for $0 \leq t < 1$.

As a second example, we consider the problem

$$\begin{cases} \epsilon y''(t) - y'(t) - a\,[y(t)]^n = 0 & (0 < t < 1) \\ y(0) - \epsilon y'(0) = 1, \ y'(1) = 0. \end{cases}$$

This boundary value problem describes the concentration of a reactant undergoing axial diffusion in an isothermal tubular flow reactor. The positive constant n is for an n^{th} order reaction, and n need not be an integer. The positive constant a is related to the ratio of reaction and flow velocities, and $\epsilon > 0$ is the ratio of diffusion and flow velocities. Fan and Bailie [12] present some computations for this problem with $a = 0.1$ and $n = \frac{1}{2}, \frac{1}{4}, 2, 3$. Let us apply Theorem 6.10 to the case $n = \frac{1}{2}$. If we let $Z(0,\epsilon) = \delta^2$, then δ satisfies the equation $\delta^2 + 0.1\,\epsilon\,\delta = 1$, and hence

$$\delta = (1 + 0.0025\epsilon^2)^{\frac{1}{2}} - 0.05\epsilon\ .$$

Then it is easy to calculate

$$Z(t,\epsilon) = (\delta - 0.05t)^2,$$

$$w(t,\epsilon) = \alpha + \beta t + 0.0025t^2 + \gamma \exp\left[\frac{t-1}{\epsilon}\right]$$

$$= Z(t) + 0.005\epsilon\,(\epsilon + t) + \gamma \exp\left[\frac{t-1}{\epsilon}\right],$$

where

$$\alpha = 1 - 0.1\epsilon + 0.005\epsilon^2,$$

$$\beta = 0.005\epsilon - 0.1\delta,$$

$$\gamma = (0.1\delta - 0.005)\epsilon - 0.005\epsilon^2.$$

The theorem then states that

$$Z(t,\epsilon) \le y(t,\epsilon) \le w(t,\epsilon) \qquad (0 \le t \le 1),$$

and $\lim_{\epsilon \to 0+} w(t,\epsilon) = \lim_{\epsilon \to 0+} Z(t,\epsilon)$ for $0 \le t < 1$. These bounds provide excellent approximations to $y(t,\epsilon)$ even for rather large ϵ. For example, if $\epsilon = 0.1$ the maximum difference between the two bounding functions is 9.95×10^{-3}, and the maximum is attained at $t = 1$. For $t < 1$ the difference is relatively constant and approximately 3×10^{-4}. Since $w(t,\epsilon)$ and $Z(t,\epsilon)$ lie between 1 and 0.893, the relative error is nearly as satisfactory as the absolute error. Thus it appears that these bounds are a useful computational tool. Unlike the usual asymptotic methods, their validity and degree of approximation can be readily assessed.

7. Nonlinear Systems

In this section we turn to some special examples of pairs of second order equations of the type studied in [9].

Our first example completes the discussion of an interesting example treated in [9]. Consider the problem

$$(7.1)\quad \begin{cases} u'' = v & (0 < t < 1) \\ \epsilon v'' + u'v' = 0 & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ v(0) = A, \quad v(1) = B. & \end{cases}$$

The results of [9] resolve the cases when A and B are of the same sign. For the moment, we assume that

$$(7.2)\qquad A < 0 < B\ .$$

We now collect some basic facts.

Lemma 7.1 For each $\epsilon > 0$ there is a solution pair $\{u(t,\epsilon),\ v(t,\epsilon)\}$ of (7.1), and

$$(7.3)\qquad v'(t,\epsilon) > 0.$$

Accordingly, there is exactly one point $\alpha = \alpha(\epsilon) \in (0,1)$ such that

$$
(7.4)\quad \begin{cases} v(\alpha,\epsilon) = 0, & \\ u''(t,\epsilon) = v(t,\epsilon) < 0 & (0 \le t < \alpha) \\ u''(t,\epsilon) = v(t,\epsilon) > 0 & (\alpha < t \le 1). \end{cases}
$$

Finally,

$$
(7.5)\qquad u'(\alpha,\epsilon) = \min_{0\le t\le 1} u'(t,\epsilon) < 0.
$$

Proof: The existence follows from a fixed point argument as in [9, Theorem 1]. Inequalities (7.3) and (7.4) follow from the maximum principle. Finally, (7.5) follows from (7.4) and the fact that

$$
u'''(\alpha,\epsilon) = v'(\alpha,\epsilon) > 0,
$$

which implies that $u'(t)$ decreases near $t = 0$ and increases near $t = 1$.

Lemma 7.2. There is a sequence $\epsilon_n \to 0+$ and a pair of functions $U(t) \in C^1[0,1]$, $V(t) \in L^1[0,1]$ such that

$$
(7.6a)\qquad \lim_{\epsilon_n \to 0+} \operatorname{Max} \{ |u(t,\epsilon_n) - U(t)| + |u'(t,\epsilon_n) - U'(t)| \} = 0
$$

and, at each point $t \in [0,1]$,

$$
(7.6b)\qquad \lim_{\epsilon_n \to 0+} |v(t,\epsilon_n) - V(t)| = 0.
$$

The function $V(t)$ is monotone nondecreasing and

$$(7.7a) \qquad \begin{cases} U''(t) = V(t) & \text{a.e.} \quad (0 \leq t \leq 1) \\ U(0) = U(1) = 0. \end{cases}$$

Moreover, in any interval $(a,b) \subset [0,1]$ for which $|U'(t)| > 0$ we have

$$(7.7b) \qquad V(t) \equiv \text{constant} \qquad (a < t < b).$$

Proof: See [9 , Theorem 2].

Lemma 7.3 Suppose the sequence $\{\epsilon_n\}$ and the limit functions $U(t)$, $V(t)$ are such that (7.6a), (7.6b), (7.7a), (7.7b) hold. If

$$(7.8) \qquad U(t) \not\equiv 0 \qquad (0 \leq t \leq 1),$$

then

$$(7.9a) \qquad V(t) \leq 0 \qquad (0 < t < 1)$$

and hence

$$(7.9b) \qquad \begin{cases} U(t) > 0 & (0 < t < 1), \\ U'(0) > 0, \\ U'(1) < 0. \end{cases}$$

Proof: Once (7.9a) has been established, (7.9b) follows easily from (7.7a), the maximum principle, and the formulae

$$(7.10)\qquad \begin{cases} U'(0) = -\displaystyle\int_0^1 (1-s)\, V(s)\, ds\,, \\ U'(1) = \displaystyle\int_0^1 s\, V(s)\, ds. \end{cases}$$

Without loss of generality we can assume that

$$(7.11)\qquad \lim_{\epsilon_n \to 0+} \alpha(\epsilon) = \bar{\alpha} \in [0,1].$$

There are three possibilities.

<u>Case 1:</u> $\bar{\alpha} = 0$ and $V(t) \geq 0$ for $0 < t < 1$.

Using (7.6a) and (7.5) we see that $U'(0) \leq 0$. If $U'(0) = 0$, then

$$U'(t) = \int_0^t V(s)\, ds \geq 0$$

and

$$U(1) = \int_0^1 (1-s)\, V(s)\, ds > 0.$$

On the other hand, suppose $U'(0) < 0$.

Then we can apply the argument of Lemma 3.2 as in [9 , Theorem 3] to see that

$$V(t) = A \qquad (0 \leq t < \delta)$$

for some $\delta > 0$. Since this is impossible, we see that

$$\bar{\alpha} \neq 0 \,. \tag{7.12}$$

Case 2: $0 < \bar{\alpha} < 1$. Then we have

$$\begin{cases} U'(\bar{\alpha}) < 0 & \\ V(t) \geq 0 & (t > \bar{\alpha}), \\ V(t) \leq 0 & (t < \bar{\alpha}). \end{cases} \tag{7.13}$$

Thus, using Lemma 7.2 we see that there is a $\Delta > 0$ such that

$$V(t) = 0 \qquad (\bar{\alpha} \leq t < \bar{\alpha} + \Delta).$$

We choose Δ as large as possible. Either $\bar{\alpha} + \Delta = 1$ and the lemma follows, or

$$U'(\bar{\alpha} + \Delta) = 0 \,.$$

However,

$$U'(\bar{\alpha} + \Delta) = U'(\bar{\alpha}) + \int_{\bar{\alpha}}^{\bar{\alpha}+\Delta} V(t)\, dt = U'(\bar{\alpha}) < 0.$$

Hence, in this case, the conclusion of the lemma is established.

Case 3: $\bar{\alpha} = 1$. Then

$$V(t) \leq 0 \qquad (t < 1),$$

and the lemma is proven.

Let

$$(7.14) \qquad G(s,\epsilon) = \exp\{-\frac{1}{\epsilon} u(s,\epsilon)\}.$$

Our next lemmas are merely restatements of some results of [9].

Lemma 7.4. For all $\epsilon > 0$

$$(7.15) \qquad v(\tfrac{1}{2},\epsilon) \le \tfrac{1}{2}(A+B).$$

Proof: As in [9 , Lemma 2], we have

$$(7.16) \qquad v(t,\epsilon) = A + (B-A)\left[\int_0^t G(s,\epsilon)\,ds\right]\left[\int_0^1 G(s,\epsilon)\,ds\right]^{-1}.$$

Let $F(t) = u(t,\epsilon) - u(1-t,\epsilon)$. Then $F(\frac{1}{2}) = F(1) = 0$ and $F''(t) = v(t,\epsilon) - v(1-t,\epsilon)$. Since $v(t)$ is monotone increasing,

$$F''(t) \ge 0 \qquad (\tfrac{1}{2} \le t \le 1).$$

Thus

$$u(t,\epsilon) \le u(1-t,\epsilon) \qquad (\tfrac{1}{2} \le t \le 1),$$

and

$$G(t,\epsilon) \ge G(1-t,\epsilon) \qquad (\tfrac{1}{2} \le t \le 1).$$

Inserting this result into (7.16), we obtain (7.15).

Lemma 7.5. If $0 < \epsilon \le 1$,

$$(7.17) \qquad 0 < \left[\int_0^1 G(s,\epsilon)\,ds\right]^{-1} \le \frac{M}{\epsilon},$$

where

$$M = \frac{|A|}{4}\{1 - \exp(-\frac{|A|}{4})\}^{-1} .$$

Proof: See [9 , Lemma 3].

Lemma 7.6. Suppose $\{\epsilon_n\}$, $U(t)$, $V(t)$ are as in Lemma 7.2, so that (7.6a), (7.6b), (7.7a) and (7.7b) hold. Suppose (7.8) holds. Then

$$(7.18)\quad \begin{cases} V(t) = \frac{1}{2}(A + B). & (0 < t < 1), \\ U(t) = \frac{1}{4}(A + B)(t^2 - t) & (0 \le t \le 1). \end{cases}$$

Proof: Using Lemmas 7.3 and 7.5, and the argument of [9 , Lemma 4], we obtain (7.18).

Theorem 7.1. Let $u(t,\epsilon)$ $v(t,\epsilon)$, be a solution of (7.1) subject to (7.2).

(i) If $A + B \ge 0$,

$$(7.19)\quad \lim_{\epsilon \to 0+} u(t,\epsilon) = \lim_{\epsilon \to 0+} v(t,\epsilon) = 0 \qquad (0 < t < 1).$$

(ii) If $A + B < 0$,

$$(7.20)\quad \lim_{\epsilon \to 0+} v(t,\epsilon) = \frac{1}{2}(A + B) \qquad (0 < t < 1),$$

and

$$(7.21) \qquad \lim_{\epsilon \to 0+} u(t,\epsilon) = \frac{1}{4}(A+B)(t^2 - t) \qquad (0 \le t \le 1).$$

Proof: Choose a sequence $\epsilon_n \to 0+$ so that $u(t_0, \epsilon_n)$ and $v(t_0, \epsilon_n)$ converge for some fixed $t_0 \in (0,1)$. Using the compactness, we can choose a subsequence (which we continue to call ϵ_n) so that all the hypotheses of Lemma 7.2 are satisfied.

Using Lemma 7.3, we see that either $U(t) \equiv 0$ or (7.9a) holds. Using Lemma 7.6 we see that either $U(t) \equiv 0$ or

$$V(t) = \frac{1}{2}(A+B).$$

Hence, if (i) holds, then (7.19) holds. On the other hand, if (ii) holds, then Lemma 7.4 and the monotone behavior of $v(t)$ give

$$V(t) \le \frac{1}{2}(A+B) < 0 \qquad (0 \le t \le \frac{1}{2}).$$

Thus $V(t) \not\equiv 0$ and $U(t) \not\equiv 0$, and the theorem follows from Lemma 7.6.

Remark: It is an easy matter to show that there is at least one point $\beta = \beta(\epsilon) \in (0,1)$ such that

$$u'(\beta, \epsilon) = 0 .$$

Indeed, there cannot be more than two such points. There are three cases.

Case 1: $u'(0,\epsilon) \leq 0$. Then there is only one such point β, and

$$\alpha(\epsilon) < \beta(\epsilon) .$$

Case 2: $u'(0,\epsilon) > 0$ and $u'(1,\epsilon) < 0$. Then there is only one such point β and

$$\beta(\epsilon) < \alpha(\epsilon).$$

Case 3: $u'(0,\epsilon) > 0$ and $u'(1,\epsilon) > 0$. Then there are two points β_1 and β_2, and

$$\beta_1(\epsilon) < \alpha(\epsilon) < \beta_2(\epsilon) .$$

Using Lemma 7.3 we can easily prove the following result.

Lemma. There is an $\epsilon_0 > 0$ such that $0 < \epsilon \leq \epsilon_0$ implies that Case 2 occurs.

Proof: Suppose not. Then there is a sequence $\epsilon_n \to 0+$ such that Case 1 (Case 3) occurs for all ϵ_n small enough. After extracting a convergent subsequence we have $U'(0) \leq 0$ in Case 1, and $U'(1) \geq 0$ in Case 3. In either case, this contradicts Lemma 7.3.

Remark:

In all cases, one can show that there exists an $\epsilon_0 > 0$ such that $u'(0,\epsilon) > 0$ for $0 < \epsilon \leq \epsilon_0$.

Remark:

Part of the interest in this problem is the fact that in Case 3 we are dealing with two unknown turning points β_1 and β_2 for the reduced equation.

We are now in a position to summarize the asymptotic behavior of the solutions $\{u(t,\epsilon),\ v(t,\epsilon)\}$ of (7.1). In all cases the limit functions are related by (7.7a).

Case 1: $(A + B) \leq 0$. Then

(7.22) $$\lim_{\epsilon \to 0+} v(t,\epsilon) = \frac{1}{2}(A + B) \qquad (0 < t < 1).$$

Proof: See [9 , Theorem 6] and Theorem 7.1. If $A > B$, consider $u(1 - t,\epsilon)$ and $v(1 - t,\epsilon)$.

Case 2: $A < 0$ or $B < 0$ but $A + B > 0$. Then

(7.23) $$\lim_{\epsilon \to 0+} v(t,\epsilon) = 0 \qquad (0 < t < 1).$$

Proof: Apply Theorem 7.1. Again, if $A > B$ consider $u(1 - t,\epsilon)$ and $v(1 - t,\epsilon)$.

Case 3: $0 \leq A \leq B$. Then,

$$(7.24) \qquad \lim_{\epsilon \to 0+} v(t,\epsilon) = \begin{cases} A & 0 \le t < \alpha \\ B & \alpha < t \le 1 , \end{cases}$$

where

$$(7.24a) \qquad \alpha = \frac{1}{1 + \sqrt{A/B}} .$$

Proof: See [9 , Theorem 3]

Case 4: $0 \le B \le A$. Then

$$(7.25) \qquad \lim_{\epsilon \to 0+} v(t,\epsilon) = \begin{cases} A & 0 \le t < \alpha \\ B & \alpha < t \le 1 . \end{cases}$$

Proof: Consider $u(1-t,\epsilon)$ and $v(1-t,\epsilon)$ and we have Case 3.

We now consider two other examples of the type studied in [9] .

Consider the nonlinear system

$$(7.26) \qquad \begin{cases} u''(t) = v & (0 \le t \le 1) \\ \epsilon v''(t) + u'(t)\, v'(t) - |u'(t)|\, v(t) = 0 & (0 \le t \le 1) \\ u(0) = u(1) = 0 & \\ 0 \le A = v(0) < v(1) = B . & \end{cases}$$

Once more, we know there are solutions $\{u(t,\epsilon), v(t,\epsilon)\}$ and Lemma 7.2 holds.

Theorem 7.2. Let $u(t,\epsilon)$, $v(t,\epsilon)$ be solutions of (7.26). Let μ be the unique root in $(0,1]$ of the equation

$$A[1-(1+\mu)e^{-\mu}] = B[1+(\mu-2)e^{\mu-1}]. \tag{7.27a}$$

Then

$$\lim_{\epsilon\to 0+} v(t,\epsilon) = \begin{cases} Ae^{-t} & 0 \le t < \mu \\ Be^{t-1} & \mu < t \le 1. \end{cases} \tag{7.27b}$$

Proof: Let $\epsilon_n \to 0+$ and let $U(t)$, $V(t)$ be limit functions. If $A = 0$, it follows from [9 , Theorem 4] that $V(t) \equiv 0$. If $A > 0$, by using the technique of Theorem 3.5 we see that

$$V(t) \ge \frac{A}{e} > 0 \qquad (0 \le t \le 1). \tag{7.28}$$

Hence there is a unique $\mu \in (0,1)$ such that $U'(\mu) = 0$. With the use of (7.28), it is easy to see that (7.27b) holds. Finally, we substitute (7.27b) into (7.7a). Using the fact that $U(t) \in C^1[0,1]$ and $U'(\mu) = 0$ we find that μ satisfies (7.27a).

The asymptotic behavior in the case $f(t,u,v) = -v$ can be determined from the following more general result.

Theorem 7.3. Let $u(t,\epsilon)$ and $v(t,\epsilon)$ be solutions to

$$\begin{cases} u''(t) = -v(t) & (0 < t < 1) \\ \epsilon v''(t) + u'(t)\,v'(t) - c(t,u(t),\,u'(t)\,)\,v(t) = 0 & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ 0 \le v_0 = v(0) < v(1) = v_1 \,. & \end{cases}$$

Assume that there exists a $\delta_0 > 0$ such that

$$c(t,u,u')\,c(1-t,u,u') > 0$$

for $0 < t \le \delta_0$ and for all u, u' with $u\,u' \neq 0$. Then

$$\lim_{\epsilon \to 0+} v(t,\epsilon) = 0 \qquad (0 < t < 1)\,.$$

Proof: Let $V(t)$ be a limit function, and let $U(t)$ be the solution of (7.7a). Assume that we do not have $V(t) \equiv 0$ for $0 < t < 1$. Then

$$U'(0) = \int_0^1 (1-x)\,V(x)\,dx > 0$$

and

$$U'(1) = -\int_0^1 x\,V(x)\,dx < 0\,.$$

Since $U'(t)$ is monotone non-increasing, there exists a $\delta > 0$ such that

$$\begin{cases} U'(t) \ge U'(\delta) > 0 & (0 \le t \le \delta) \\ U'(t) \le U'(1-\delta) < 0 & (1-\delta \le t \le 1)\,. \end{cases}$$

The rest of the proof of the theorem now follows easily from Theorem 3.6.

REFERENCES

1. P. B. Bailey, L. F. Shampine, and P. E. Waltman, Nonlinear Two Point Boundary Value Problems, Academic Press, New York, 1968.

2. ————————————————, "Bounding curves for unforced nonlinear boundary value problems," unpublished.

3. G. F. Carrier, "Singular perturbation theory and geophysics," SIAM Review 12 (1970), 175-193.

4. J. A. Cochran, "Problems in singular perturbation theory," Ph.D. Dissertation, Stanford University, 1962.

5. E. A. Coddington and N. Levinson, "A boundary value problem for a nonlinear differential equation with a small parameter," Proc. Amer. Math. Soc. 3 (1952), 73-81.

6. ————————————————, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

7. J. D. Cole, Perturbation Methods in Applied Mathematics, Blaisdell Publishing Co., Waltham, Mass. 1968 .

8. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I. Interscience, New York, 1953 .

9. F. W. Dorr and S. V. Parter, "Singular perturbations of nonlinear boundary value problems with turning points," J. Math. Anal. Appl., 29 (1970), 273-293.

10. F. W. Dorr, "Some examples of singular perturbation problems with turning points." SIAM J. Math. Anal., 1 (1970), 141-146.

11. W. Eckhus and E. M. DeJager, "Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type," Arch. Rational Mech. Anal. 23 (1966), 26-86.

12. Liang-Tseng Fan and Richard C. Bailie, "Axial diffusion in isothermal tubular flow reactors," Chem. Engng. Sci., 13 (1960), 63-68.

13. K. O. Friedrichs, "The identity of weak and strong extensions of differential operators," Trans. Amer. Math. Soc. 55 (1944), 132-151.

14. S. Haber and N. Levinson, "A boundary value problem for a singularly perturbed differential equation," Proc. Amer. Math. Soc. 6 (1955), 866-872.

15. N. Levinson, "The first boundary value problem for $\epsilon \Delta u + A(x,y) u_x + B(x,y) u_y + C(x,y) u = D(x,y)$ for small ϵ," Annals of Mathematics 51, (1950), 428-445.

16. R. v. Mises, "Die Grenzschichte in der Theorie der gewöhnlichn Differential gleichungen," Acta. Univ. Szeged, Leopoldo Fejer er Frederica Riesz LXX annes natis dedicatus, Pars B 12 (1950), 29-34.

17. I. P. Natanson, Theory of Functions of a Real Variable, Ungar, New York, 1955.

18. R. E. O'Malley, Jr., "Two-parameter singular perturbation problems for second-order equations," J. Math. Mech., 16 (1967), 1143-1164.

19. ____________________, "Topics in singular perturbations," Advances in Math., 2 (1968), 365-470.

20. ____________________, "A boundary value problem for certain nonlinear second order differential equations with a small parameter," Arch. Rational Mech. Anal., 29 (1968), 66-74.

21. ____________________, "On the asymptotic solution of boundary value problems for nonhomogeneous ordinary differential equations containing a parameter," J. Math. Anal. Appl., 28 (1969), 450-460.

22. ____________________, "On boundary value problems for a singularly perturbed differential equation with a turning point," to appear.

23. ____________________, "On nonlinear singular perturbation problems with interior nonuniformities," Journal Math. and Mech. 12 (1970), 1103-1112.

24. ____________________, "A nonlinear singular perturbation problem arising in the study of chemical flow reactors," Mathematics Research Center, University of Wisconsin Report, #847.

25. M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, N. J., 1967.

26. F. Riesz and B. Sz-Nagy, Functional Analysis, F. Ungar Publishing Co., New York, 1955 .

27. A. B. Vasil'eva, "Asymptotic behaviour of solutions of the problems for ordinary nonlinear differential equations with a small parameter multiplying the highest derivatives" Uspekhi Mat. Nauk, 18 (1963) no. 3, 15-86, Russian Math. Surveys 18 (1963).

28. W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Interscience, New York, 1965.

29. __________, "The Capricousness of singular perturbations." Nieuw Archief Voor Wiskunde XVIII (1970), 190-210.

30. M. Zlamal, "The parabolic equation as a limiting case of a certain elliptic equation," Ann. Mat. Pura, Appl. 57, (1962), 143-150.