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MINIMUM ERROR BOUNDS FOR MULTIDIMENSIONAL SPLINE APPROXIMATION

by

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ABSTRACT

Approximation of a smooth function f on a rectangular domain $\Omega \subset E^{\ell}$, by a tensor product of splines of degree m is considered. A basis for the product spline is formed using a single one-dimensional spline function. The approximation is computed, using linear programming, so as to minimize the maximum error on a discrete grid $\Omega_{\nu} \subset \Omega$, with grid size h. Realistic a posteriori bounds on the error in the uniform norm are given. Convergence of the approximation to a best approximation as $h \to 0$ is shown. The extension to linear boundary value problems is also discussed.

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1. INTRODUCTION

A classical problem of approximation theory is that of approximating a smooth function f on an interval by functions which depend only on a finite number of parameters. More recent work has generalized this approach to include approximation on multidimensional domains and the approximate solution of ordinary and partial differential equation boundary value problems. A useful computational implementation of such a generalized approximation problem requires both a specific algorithm and an error bound which can be computed. An important limitation which must be considered, particularly in the case of the error bound, is that the functions can only be numerically evaluated at a finite subset of points in the domain of interest. Since the computer time required will increase with the number of such points, it is important to keep this number as small as possible consistent with accuracy requirements.

The main purpose of this paper is to relate the uniform error in the approximation to the error computed on the finite set of points. This relation, together with linear programming, is then used to determine the approximation so as to minimize the bound on the error in the uniform norm. The approximating functions chosen are splines of specified degree m on an interval, and the tensor product of these for multidimensional domains.

In Section 2, known a priori error bounds, based on interpolating spline approximation, are summarized. Relations are then obtained which give a bound on the uniform error over the interval I in terms of the maximum error on the discrete grid I_ν. It is shown (Theorem 1 and corollaries) that these two errors differ in certain cases, by a term of order $h^{m+1} \|f^{(m+1)}\|$, where h is the discrete grid size. Thus for smooth f and reasonable values of h the error in the approximation is shown to be essentially that attained on the discrete grid I_ν. Numerical comparison for certain cases given in Table 1 (Section 6) shows that the error bound given by Theorem 1 may be orders of magnitude smaller than the a priori error bounds.

Corresponding results for approximation by a tensor product of splines on a rectangular domain Ω in ℓ-dimensional Euclidean space are obtained in Section 3. A discrete grid $\Omega_\nu \subset \Omega$ is considered, where h is again a measure of the grid size. A result, similar to that for the one-dimensional case, giving a bound on the uniform error over Ω is given in Theorem 2 and Corollaries 3.1 and 3.2. The term depending on h^{m+1} is now proportional to terms involving the derivative of f with respect to each independent variable. This is in contrast to a priori bounds which depend on essentially all of the mixed partial derivatives (see for example [11]). As a result it is usually a simple matter to choose the grid Ω_ν so that the error bound is given essentially by the error on Ω_ν. It follows directly from the error bound that the minimum error bound is achieved by minimizing the error on the discrete grid Ω_ν. This is given as Corollary 3.3.

The question of convergence to the best approximation as the grid size approaches zero is considered in Section 4. This is done in terms of approximation by a generalized polynomial $V(\alpha, x)$, with n coefficients represented by the vector α. Assuming that $\alpha = \alpha^\nu$, is determined so as to minimize the error on each Ω_ν, convergence is shown (Theorem 3) as the grid size approaches zero ($\nu \to \infty$), with fixed n. In Corollary 4.1 it is shown that one can obtain as accurate a bound on the uniform error as desired by evaluation only (as distinct from minimization) over a finer grid.

A finite dimensional basis for product splines of degree m with uniform knot size is described in Section 5. This basis is formed from a single spline $\beta_m(\tau)$, with compact support on an interval. This representation simplifies the computation and leads to well-conditioned matrices.

In Section 6 the formulation as a linear programming problem is given. This formulation permits the efficient calculation of the product spline approximation and its error bound by solving a primal linear programming problem with $n + 1$ rows. Numerical results for approximation on an interval and on a rectangular domain are presented. Error bounds are given for these examples, and compared with corresponding a priori error bounds in the literature (Tables 1 and 2).

Finally, in Section 7, the earlier results are extended in a natural manner to the approximate solution by splines of certain linear boundary value problems. It is shown that the approximate solution with a minimum error bound may be obtained by solving a single linear programming problem.

2. APPROXIMATION ON AN INTERVAL

We first consider the (one-dimensional) approximation of a given function $f(x)$ on the closed interval $I = [0,1]$, by a suitable spline function. Let $s_m(\Delta;x)$ denote a spline of degree m with maximum knot size Δ. We assume that $f \in C^{m+1}[I]$, and consider the uniform norm $\|\cdot\| = \sup_I |\cdot|$. A number of papers (for example, see [12, 7, 11]) have considered a priori bounds on the error in $\tilde{s}_m$, where $\tilde{s}_m$ is the spline which interpolates f at the knots. These bounds are all of the general form

$$(2.1) \qquad \|f - \tilde{s}_m\| \leq k_m \Delta^{m+1} \|f^{(m+1)}\|$$

where k_m is a constant independent of Δ and f.

A more general approach, which includes interpolation as a special case, is the construction of a discrete grid of $\nu + 1$ points $I_\nu \subset I$, such that all knots in I are contained in I_ν. The grid I_ν is assumed to be dense in I as $\nu \to \infty$. The grid density is measured by $|I_\nu|$, where

$$(2.2) \qquad |I_\nu| = \max_{x \in I} \min_{y \in I_\nu} |x - y|$$

To simplify the notation let

$$(2.3) \qquad h = h(\nu) = 2\,|I_\nu|$$

A uniform grid gives $h = \nu^{-1}$, that is, h is the grid size. The discrete seminorm corresponding to $\|\cdot\|$ is then given by $\|\cdot\|_h = \max_{I_\nu} |\cdot|$.

The problem of finding $s_m(\Delta, x)$ so that $\|f - s_m\|_h = \min$, is computationally a nice one, and can be solved efficiently using linear programming, as discussed in Section 6. Note that if I_ν consists of just the knots (in I) of $s_m(\Delta, x)$, then this minimization problem gives the interpolation spline with $\|f - s_m\|_h = 0$. The general question to be considered in this section is the relation between $\|f - s_m\|$ and $\|f - s_m\|_h$ for $h < \Delta$.

This relation has been considered for example by Cheney [3], assuming only that $f \in \text{Lip}(\lambda)$. A bound of the following kind is then obtained

$$\|f - s_m\| \leq \|f - s_m\|_h + \frac{h}{2}[\lambda + k_m \|f\|] \tag{2.4}$$

where k_m is a constant independent of h and f. For any fixed spline $s_m(\Delta, x)$, it follows from the definition of $\|\cdot\|_h$ and $I_\nu \subset I$ that $\|f - s_m\|_h \leq \|f - s_m\|$. Therefore as $h \to 0$ the discrete error $\|f - s_m\|_h$ converges to the uniform error $\|f - s_m\|$.

Denote by $\hat{s}_m(h)$ the spline approximation obtained by minimizing $\|f - s_m\|_h$. It is found computationally that for smooth f and $h < \frac{\Delta}{2}$, $\hat{s}_m(h)$ is a better approximation than the interpolating spline $\tilde{s}_m$. That is, $\|f - \hat{s}_m(h)\| < \|f - \tilde{s}_m\|$ for $h < \frac{\Delta}{2}$. For example, see the computational results in [4] and in Table 1 (Section 6). One would expect that the error bounds (2.1) and (2.4) would reflect this observed behavior, especially since (2.4) is an a posteriori bound. In fact, the opposite is found to hold generally. That is, for most functions $f \in C^{m+1}$ the second term on the right-hand-side of (2.4) is much larger than the right-hand-side of (2.1), unless $h << \Delta$. Therefore for reasonable h, (2.4) gives a very unrealistic bound for the uniform error $\|f - \hat{s}_m(h)\|$, even though the value of $\|f - \hat{s}_m(h)\|_h$ is known. In order to do better, we must use the smoothness of f and get an a posteriori bound with higher order convergence than the linear dependence on h given by (2.4).

A relatively simple problem is basic to much of what follows. We consider the approximation of a given function $f \in C^{m+1}$ on the interval $X = [0,\Delta]$ by a polynomial $P_m(x)$ of degree m. We choose a uniform grid $X_q \subset X$ of $q + 1$ points, $x_j = jh$, $j = 0,1,\ldots q$, where $qh = \Delta$, and $q \geq \max\{2,m\}$. Now let $\|\cdot\| = \max_X |\cdot|$, and

$$\|\cdot\|_h = \max_{X_q} |\cdot|.$$

We also need the norm λ_m of the Lagrangian interpolating polynomial of degree m. Let $\ell_{m,i}(x)$ be the Lagrangian interpolating polynomials of degree m, for the $m+1$ points $x_j = j$, $j = 0,1,\ldots m$, so that $\ell_{m,i}(x_j) = \delta_{ij}$. Then

$$Q_m(g;x) = \sum_{i=0}^{m} g(x_i)\ell_{m,i}(x) \tag{2.5}$$

is the unique m^{th} degree polynomial interpolating g at the points x_j. Now if

$$\lambda_m = \max_{x\in[o,m]} \sum_{i=0}^{m} |\ell_{m,i}(x)| \tag{2.6}$$

then

$$\max_{x\in[o,m]} |Q_m(g;x)| \leq \lambda_m \max_j |g(x_j)| \tag{2.7}$$

The values of λ_m are easily computed and for $m = 1,2,\ldots 5$, are given approximately by $\lambda_1 = 1.00$, $\lambda_2 = 1.25$, $\lambda_3 = 1.63$, $\lambda_4 = 2.21$ and $\lambda_5 = 3.11$. The corresponding approximate values of $\bar{\lambda}_m$ as given by (2.9) below are $\bar{\lambda}_1 = 0.5$, $\bar{\lambda}_2 = 1.67$, $\bar{\lambda}_3 = 5.5$, $\bar{\lambda}_4 = 18.8$, $\bar{\lambda}_5 = 67.5$.

For any polynomial $P_m(x)$ on X, the relation between $\|f - P_m\|_h$ and $\|f - P_m\|$ is then given by

<u>Theorem 1</u>

Assume $f \in C^{m+1}[X]$, and X_q as described above. Then

$$(2.8) \qquad \|f - P_m\| \le \begin{cases} \lambda_m \|f - P_m\|_h + \bar{\lambda}_m h^{m+1} \|f^{(m+1)}\|, & h \le \Delta/m \\ \min\limits_{0 \le \theta \le 1} E_m(f, P_m, h, \theta), & h \le \Delta/m^2 \end{cases}$$

where

$$\bar{\lambda}_m = \frac{m^{m+1}}{(m+1)!} \lambda_m$$

$$E_m(f, P_m, h, \theta) = [1 + \eta_m(h,\theta)h^{\theta}] \|f - P_m\|_h + \omega_m(h,\theta)h^{m+1} \|f^{(m+1)}\|$$

(2.9)

$$\eta_m(h,\theta) = [\frac{5}{m^2}(\Delta - m^2 h)^{\theta} + 5h^{\theta}]^{-1}$$

$$\omega_m(h,\theta) = \frac{1.2}{(m+1)!} (m^2 + \Delta^{\theta} h^{-\theta})^{m+1}$$

<u>Proof</u>: To obtain the first inequality of (2.8) we let $\rho(x) = f(x) - P_m(x)$. Assume that $|\rho(x)|$ attains its maximum value $\|\rho\|$ at $x = \bar{x} \in X$, that is $\pm\rho(\bar{x}) = \|\rho\|$. Then by a Taylor's expansion about $\bar{x}$, we have for $x \in X$,

$$(2.10) \qquad \rho(x) = p_m(x) \pm \frac{1}{(m+1)!} \rho^{(m+1)}(\hat{x}) (x - \bar{x})^{m+1}$$

where $p_m(x)$ is a polynomial of degree m, and $\hat{x}$ is some point in X. Now since $P_m^{(m+1)} = 0$, we have

$$(2.11)\qquad |p_m(x)| \le |\rho(x)| + \frac{|x-\bar{x}|^{m+1}}{(m+1)!} \quad \|f^{(m+1)}\|$$

In particular for $x_j \in X_q$

$$(2.12)\qquad |p_m(x_j)| \le \|\rho\|_h + \frac{|x_j-\bar{x}|^{m+1}}{(m+1)!} \quad \|f^{(m+1)}\|$$

Let $X_m \subset X_q$ denote a set of $m+1$ points, say $x_k, x_{k+1}, \ldots, x_{k+m}$, such that $\bar{x} \in [x_k, x_{k+m}]$. Then for $x_j \in X_m$, we have $|x_m - \bar{x}| \le mh$, so that

$$(2.13)\qquad |p_m(x_j)| \le \|\rho\|_h + \frac{m^{m+1}}{(m+1)!} \quad h^{m+1} \; \|f^{(m+1)}\|$$

Now $p_m(x)$ is the unique polynomial of degree m which interpolates the $m+1$ values $p_m(x_j)$, $x_j \in X_m$. Then by (2.7), since $\bar{x} \in [x_k, x_{k+m}]$,

$$(2.14)\qquad |p_m(\bar{x})| \le \|p_m(x)\| \le \lambda_m \max_{x_j \in X_m} |p_m(x_j)|$$

From (2.10) we have $\|\rho\| = \rho(\bar{x}) = p_m(\bar{x})$, so that (2.13) and (2.14) give the first inequality of (2.8).

To get the second inequality, we assume $h \le \frac{\Delta}{m^2}$, so that $q \ge m^2$ and let n be an integer, $m^2 \le n \le q$, and consider a subset $X_n \subset X_q$ of $n+1$ points, say $x_k, x_{k+1}, \ldots, x_{k+n}$, with $\bar{x} \in [x_k, x_{k+n}]$.

We wish to use Lemma 2b of [9] on the interval $[x_k, x_{k+n}]$. The lemma is modified slightly to take account of the interval of length nh instead of 2. Also we let $\|p_m\|_h = \max_{x_j \in X_n} |p_m(x_j)|$. We then obtain the bound

$$(2.15) \qquad \|p_m\| \le \gamma_{m,n} \|p_m\|_h$$

where

$$(2.16) \qquad \gamma_{m,n} = [1 - \frac{m^2(m^2-1)}{6n^2}]^{-1} \le 1 + \frac{m^4}{5n^2}$$

Again using (2.12) we have

$$(2.17) \qquad \|p_m\|_h \le \|\rho\|_h + \frac{(nh)^{m+1}}{(m+1)!} \|f^{(m+1)}\|$$

Combining these two gives

$$(2.18) \qquad \|\rho\| \le \gamma_{m,n}\|\rho\|_h + \gamma_{m,n}\frac{(nh)^{m+1}}{(m+1)!} \|f^{(m+1)}\|$$

In order that $\gamma_{m,n} \to 1$ for fixed m as $h \to 0$ $(q \to \infty)$, we require $n \to \infty$. At the same time we require $nh \to 0$. A suitable choice is $n = \{m^2 + (q-m^2)^\theta\}$ where $0 < \theta < 1$; and $\{x\}$ = largest integer in x. Since $\Delta = qh \ge m^2 h$, for any selected θ, we have

$$nh \le (m^2 + q^\theta)h = (\Delta^\theta + m^2h^\theta)h^{1-\theta}$$

and

$$\gamma_{m,n} \le 1 + [\frac{5}{m^2}(\Delta - m^2 h)^\theta + 5h^\theta]^{-1} h^\theta \le 1.2$$

Using these inequalities in (2.18) for any fixed θ, gives $\|f - P_m\| \le E_m(f, P_m, h, \theta)$. Since this holds for each θ, it holds for the choice of θ which achieves the minimum for any fixed Δ, m and $h \le \frac{\Delta}{m^2}$. ∎

Now consider the approximation of a given $f(x)$ on the interval $I = [0,1]$ by a spline $s_m(\Delta;x)$ with uniform knot size Δ. We assume knots at the end points $x = 0, 1$, so that Δ^{-1} is an integer. Denote the knots by $I_\Delta \subset I$. A uniform grid I, with $I_\Delta \subset I_\nu \subset I$, of size $h = \nu^{-1}$ is constructed, with $h = \Delta/q$, where $q \ge \max\{2, m\}$. A bound on the uniform error $\|f - s_m\|$ is now given by

Corollary 2.1

Assume $f \in C^{m+1}[I]$. Then the error bound of Theorem 1 holds with $f - s_m$ replacing $f - P_m$, and $\|\cdot\|_h$ representing the maximum error on I_ν.

Proof: Since $I_\Delta \subset I_\nu$, the maximum value of $|f - s_m|$ at the knots is bounded by $\|f - s_m\|_h$, so the bound (2.8) certainly holds on I_Δ. In the intervals of length Δ between the knots, s_m is a polynomial of degree m so that Theorem 1 holds. ∎

The requirement that I_ν be a uniform grid can be eliminated by using only the second inequality in (2.8).

Corollary 2.2

Assume that $|I_\nu| = h/2 \le \Delta/2m^2$, and that $f \in C^{m+1}[I]$. Then the second inequality of (2.8) holds with $f - s_m$ replacing $f - P_m$, and $\|\cdot\|_h$ representing the maximum error on I_ν.

Proof: Lemma 2b of [9] holds for a nonuniform grid I_ν, so that the second inequality of (2.8) is valid on $I - I_\Delta$. ∎

Finally an asymptotic bound for $\|f - s_m\|$ as $h \to 0$ $(\nu \to \infty)$ follows directly from (2.8) and (2.9).

Corollary 2.3

As $h \to 0$ we have

$$(2.19) \qquad \|f - s_m\| \le \min_{0 \le \theta \le 1} \tilde{E}_m(f, s_m, h, \theta)$$

where

$$(2.20) \qquad \tilde{E}_m(f, s_m, h, \theta) = \left[1 + \frac{m^2}{5}\left(\frac{h}{\Delta}\right)^{\theta}\right]\|f - s_m\|_h + \frac{1.2}{(m+1)!}\left(\Delta^{\theta} h^{1-\theta}\right)^{m+1}\|f^{(m+1)}\| \quad ∎$$

These bounds are illustrated with numerical results in Section 6.

3. APPROXIMATION ON A RECTANGULAR DOMAIN

The results of the previous section for an interval will now be extended to a rectangular domain in ℓ-dimensional Euclidean space. We consider approximation by a tensor product of splines on this rectangular domain. This more general problem has also been considered from the point of view of interpolation (see for example [1,2]).

The main result is given in terms of the simpler problem of approximation by a product of polynomials of degree m, on a square domain. Let $G \subset E^{\ell}$ denote a square domain with sides of length Δ, and coordinates x_j, $i = 1, \cdots \ell$. Thus $G = \{x | x_j \in [0,\Delta]$, $j = 1, \cdots \ell\}$. Each coordinate is subdivided by a uniform grid of $q + 1$ points, $q \geq \max\{2,m\}$, where $qh = \Delta$, and where the grid includes the end points $x_j = 0,\Delta$. We denote by G_q the square grid of $(q+1)^{\ell}$ points obtained in this way. The norms $\|\cdot\|$ and $\|\cdot\|_h$ will denote the maximum on G and G_q respectively.

Given the function $f \in C^{m+1}[G]$, we consider its approximation by $P_m(x) = \prod_j P_{m,j}(x_j)$, where the $P_{m,j}(x_j)$ are polynomials of degree m in x_j, for $x_j \in [0,\Delta]$. Also let $D^m_{x_j}$ denote the m^{th} partial derivative with respect to the variable x_j. Note that $D^{m+1}_{x_j} P_m(x) = 0$, $j = 1, \cdots \ell$.

<u>Theorem 2</u>

Assume that

$$(3.1) \qquad \| D_{x_j}^{m+1} f \| \le \sigma_j \ , \quad j = 1, \cdots \ell$$

with $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_\ell$. Then

$$(3.2) \qquad \| f - P_m \| \le \min_{0 \le \theta \le 1} \{ \lambda_m^\ell (h,\theta) \| f - P_m \|_h + \bar{\lambda}_m (h,\theta) h^{m+1} \times \sum_{k=0}^{\ell-1} \sigma_{\ell-k} \lambda_m^k (h,\theta) \}$$

where

$$(3.3) \qquad \lambda_m(h,\theta) = \begin{cases} \lambda_m , & h \le \Delta/m \\ 1 + \eta_m(h,\theta) h^\theta , & h \le \Delta/m^2 \end{cases}$$

and

$$(3.4) \qquad \bar{\lambda}_m(h,\theta) = \begin{cases} \lambda_m \ m^{m+1}/(m+1)! , & h \le \Delta/m \\ \dfrac{1 \cdot 2}{(m+1)!} (m^2 + \Delta^\theta h^{-\theta})^{m+1} , & h \le \Delta/m^2 \end{cases}$$

Proof:

Again let $\rho(x) = f(x) - P_m(x)$, and assume that $\|\rho\| = |\rho(\bar{x})|$, with $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_\ell)$. Also let H_j denote the $q + 1$ grid points corresponding to the variable x_j, $j = 1, \ldots \ell$. It follows that

$$(3.5) \qquad \|\rho\|_h = \max_{x_j \in H_j} |\rho(x_1, x_2, \ldots x_\ell)|.$$

Consider the $(q + 1)^{\ell - t}$ points $(\bar{x}_1, \ldots, \bar{x}_t, y_{t+1}, \ldots y_\ell)$ with $y_j \in H_j$, $j = t + 1, \ldots \ell$. Assume we know a bound on the maximum of $|\rho(x)|$ over these points. That is

$$(3.6) \qquad \max_{y_j \in H_j} |\rho(\bar{x}_1, \ldots \bar{x}_t, y_{t+1}, \ldots y_\ell)| \le \xi_t$$

Now consider any specific selection $\hat{y}_j \in H_j$, $j = t + 2, \ldots \ell$. We wish to obtain a bound on $|\rho(\bar{x}_1, \ldots \bar{x}_t, x_{t+1}, \hat{y}_{t+2}, \ldots \hat{y}_\ell)|$, for $x_{t+1} \in [0, \Delta]$. Since all variables are held fixed except x_{t+1}, the one-dimensional theory can now be applied. We use the bound of Theorem 1, where now $\|f^{(m+1)}\| = \|D^{m+1}_{x_{t+1}} f\| \le \sigma_{t+1}$. We have

$$(3.7) \qquad \max_{y_{t+1} \in H_{t+1}} |\rho(\bar{x}_1, \ldots \bar{x}_t, y_{t+1}, \hat{y}_{t+2}, \ldots \hat{y}_\ell)| \le \xi_t$$

so that for $h \le \Delta/m$, the first inequality of (2.8) gives

$$(3.8) \qquad |\rho(\bar{x}_1, \ldots \bar{x}_{t+1}, \hat{y}_{t+2}, \ldots \hat{y}_\ell)| \le$$

$$\max_{x_{t+1} \in [0, \Delta]} | \rho(\bar{x}_1, \ldots \bar{x}_t, x_{t+1}, \hat{y}_{t+2}, \ldots \hat{y}_\ell)| \le \lambda_m \xi_t + \bar{\lambda}_m h^{m+1} \sigma_{t+1}$$

Since this holds for every choice $\hat{y}_j \in H_j$, $j = t + 2, \ldots \ell$, we must have

(3.9) $$\max_{y_j \in H_j} |\rho(\bar{x}_1, \cdots \bar{x}_{t+1}, y_{t+2}, \cdots y_\ell)| \leq \xi_{t+1}$$

where ξ_{t+1} satisfies the recursion relation

(3.10) $$\xi_{t+1} = \lambda_m \xi_t + \bar{\lambda}_m h^{m+1} \sigma_{t+1}$$

Starting with $t = 0$, this recursion has the solution

(3.11) $$\xi_\ell = \lambda_m^\ell \xi_0 + \bar{\lambda}_m h^{m+1} \sum_{k=0}^{\ell-1} \sigma_{\ell-k} \lambda_m^k$$

From (3.9) with $t+1 = \ell$, we have $|\rho(\bar{x})| \leq \xi_\ell$. By (3.5) we choose $\xi_0 = \|\rho\|_h$, so that

(3.12) $$\|\rho\| = |\rho(\bar{x})| \leq \lambda_m^\ell \|\rho\|_h + \bar{\lambda}_m h^{m+1} \sum_{k=0}^{\ell=1} \sigma_{\ell-k} \lambda_m^k$$

This is just (3.2) for $h \leq \Delta/m$.

In a similar way we obtain (3.2) for $h \leq \Delta/m^2$, using the corresponding bound from (2.8). ∎

In order to extend this bound to the approximation of $f(x)$ on a rectangular domain we consider the domain $\Omega \subset E^\ell$,

(3.13) $$\Omega = \{x \mid 0 \leq x_j \leq b_j, \; j = 1, \cdots \ell\}$$

Let $S_m(\Delta, x)$ denote a tensor product of one-dimensional splines of degree m on Ω, with uniform knot size $\Delta_j \leq \Delta$ corresponding to the variable x_j. The knot sizes are determined so that for positive integers μ_j, $j = 1, \cdots \ell$, we have $\mu_j \Delta_j = b_j$. Specifically we consider

$$(3.14) \qquad S_m(\Delta, x) = \prod_{j=1}^{\ell} S_m(\Delta_j, x_j)$$

where $S_m(\Delta_j, x_j)$ has knots at $x_j = 0, \Delta_j, 2\Delta_j, \cdots b_j$, and $\Delta = \max_j \Delta_j$. Let Ω_Δ denote the grid of $\prod_{j=1}^{\ell} (\mu_j+1)$ points formed by the knots in this way.

A finer grid Ω_ν, with $\Omega_\Delta \subset \Omega_\nu$ is constructed by further division of the coordinate x_j into uniform intervals of length $h_j = \Delta_j/q$, where $q \geq \max\{2, m\}$. Also let $h = \Delta/q = \max_j h_j$. A bound on the uniform error $\|f - S_m\|$ on Ω is now given by

Corollary 3.1

Assume $f \in C^{m+1}[\Omega]$. Then the error bound of Theorem 2 holds with $f - S_m$ replacing $f - P_m$, and $\|\cdot\|_h$ representing the maximum error on Ω_ν.

Proof: In each rectangular domain (with sides Δ_j) determined by Ω_Δ the spline $S_m(\Delta, x)$ is a polynomial of the form $P_m(x)$. Theorem 2 then applies to each such domain and therefore to the entire domain Ω. ∎

As in the one-dimensional case the requirement that the grid Ω_ν be uniform can be relaxed. Let I_{ν_j} denote a one-dimensional grid along the coordinate x_j, such that $I_{\nu_j} \supset I_\Delta$. The finer grid Ω_ν, with $\Omega_\Delta \subset \Omega_\nu \subset \Omega$, is then given by $\Omega_\nu = \bigcap_{j=1}^{\ell} I_{\nu_j}$. As with Corollary 2.2 we now have

Corollary 3.2

Assume that $|I_{\nu_j}| \leq \frac{h}{2} \leq \Delta/m^2$, $j = 1, \cdots \ell$, and that $f \in C^{m+1}[\Omega]$. Then the bound (3.2) for $h \leq \Delta/m^2$ holds with $f - S_m$ replacing $f - P_m$ and $\|\cdot\|_h$ representing the maximum error on Ω_ν. ∎

Corollary 3.3

For any specific function $f \in C^{m+1}[\Omega]$ assume the degree m, the knot sizes Δ_j, and the grid Ω_ν are selected. Then the minimum error bound for $\|f - S_m\|$ is achieved by determining $S_m(\Delta, x)$ so that $\|f - S_m\|_h = \min$.

Proof: For any specific f and fixed m, Δ_j and Ω_ν, the second term in the error bound (3.2) is independent of the polynomial $P_m(x)$ used. Similarly in Corollaries 3.1 and 3.2 the only term depending on the spline $S_m(\Delta, x)$ is $\|f - S_m\|_h$. ∎

4. CONVERGENCE

We will consider the question of convergence in the somewhat more general context of approximation on a closed and bounded domain $\Omega \subset E^{\ell}$, by a generalized polynomial of the form

$$(4.1) \qquad V(\alpha, x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x)$$

where the $\varphi_i(x)$ are appropriately selected functions on Ω. We wish to approximate a given function $f(x)$ on Ω by $V(\alpha, x)$ for some fixed n.

Let $\{\Omega_\nu\}$ denote a sequence of grids such that $\{\Omega_\nu\} \subset \Omega$ and $\Omega_\nu \subset \Omega_{\nu+1}$. Since $|\Omega_\nu| = \max_{x \in \Omega} \min_{y \in \Omega_\nu} |x - y|$, it follows that $|\Omega_{\nu+1}| \leq |\Omega_\nu|$. We assume that $\lim_{\nu \to \infty} |\Omega_\nu| = 0$. The norms $\|\cdot\| = \max_{\Omega} |\cdot|$ and $\|\cdot\|_\nu = \max_{\Omega_\nu} |\cdot|$ will be used. Also for any specific f, let

$$(4.2) \qquad \begin{aligned} \Psi(\alpha) &= \|f - V(\alpha)\| \\ \Psi_\nu(\alpha) &= \|f - V(\alpha)\|_\nu \end{aligned}$$

We assume that f and the φ_i are such that a best approximation $v(\alpha^*, x)$ exists. That is

$$(4.3) \qquad \Psi(\alpha^*) = \inf_{\alpha} \Psi(\alpha)$$

Let $A \subset E^n$ be a compact set containing α^*, and let $\alpha^\nu \in A$ achieve a minimum of $\Psi_\nu(\alpha)$ for $\alpha \in A$. That is

$$(4.4) \qquad \Psi_\nu(\alpha^\nu) = \min_{\alpha \in A} \Psi_\nu(\alpha)$$

Finally, we assume that for any $\alpha \in A$

$$(4.5) \qquad \Psi(\alpha) \leq \gamma_\nu \Psi_\nu(\alpha) + \omega_\nu$$

where $\gamma_\nu \rightarrow 1$ and $\omega_\nu \rightarrow 0$ as $\nu \rightarrow \infty$.

The quantities actually computed (by linear programming) are α^ν and $\Psi_\nu(\alpha^\nu)$. Convergence of $\Psi_\nu(\alpha^\nu)$ to $\Psi(\alpha^*)$ as $\nu \rightarrow \infty$ with n fixed, is given by

Theorem 3.

The sequence $\{\Psi_\nu(\alpha^\nu)\}$ converges monotonically upward to $\Psi(\alpha^*)$. For any ν the following bounds hold for $\Psi(\alpha^*)$ and $\Psi(\alpha^\nu)$,

$$(4.6) \qquad \Psi_\nu(\alpha^\nu) \leq \Psi(\alpha^*) \leq \Psi(\alpha^\nu) \leq \gamma_\nu \Psi_\nu(\alpha^\nu) + \omega_\nu$$

Proof:

Consider two grids Ω_ν and $\Omega_{\nu+1}$. Since $\Omega_\nu \subset \Omega_{\nu+1}$ we have $\Psi_\nu(\alpha^{\nu+1}) \leq \Psi_{\nu+1}(\alpha^{\nu+1})$. Also since $\Psi_\nu(\alpha)$ attains its minimum over $\alpha \in A$ for $\alpha = \alpha^\nu$, we have $\Psi_\nu(\alpha^\nu) \leq \Psi_\nu(\alpha^{\nu+1})$. Therefore $\Psi_\nu(\alpha^\nu) \leq \Psi_{\nu+1}(\alpha^{\nu+1})$, so that $\{\Psi_\nu(\alpha^\nu)\}$ is monotone increasing.

Since $\Omega_\nu \subset \Omega$ we have $\Psi_\nu(\alpha^*) \leq \Psi(\alpha^*)$. Then by (4.4) and the fact that $\alpha^* \in A$, we have

$$(4.7) \qquad \Psi_\nu(\alpha^\nu) \leq \Psi_\nu(\alpha^*) \leq \Psi(\alpha^*)$$

By (4.3) we have $\Psi(\alpha^*) \leq \Psi(\alpha^\nu)$, and using (4.5)

$$(4.8) \qquad \Psi(\alpha^*) \leq \Psi(\alpha^\nu) \leq \gamma_\nu \Psi_\nu(\alpha^\nu) + \omega_\nu$$

Combining (4.7) and (4.8) gives (4.6). It follows immediately from the monotone property of $\{\Psi_\nu(\alpha^\nu)\}$, the fact that $\gamma_\nu \to 1$, $\omega_\nu \to 0$ as $\nu \to \infty$, and (4.6) that $\Psi_\nu(\alpha^\nu) \uparrow \Psi(\alpha^*)$. ∎

This theorem will be used in the next section to show the convergence of the spline approximation (Theorem 4).

We conclude this section by showing that once a coefficient vector α^ν has been determined by minimization on Ω_ν, an improved bound on $\Psi(\alpha^\nu) = \|f - V(\alpha^\nu)\|$ can be obtained by evaluation only, using a finer grid $\Omega_{\bar{\nu}}$. Note that from (4.2) we need only evaluate $f(x) - V(\alpha^\nu, x)$ for $x \in \Omega_{\bar{\nu}}$ in order to obtain $\Psi_{\bar{\nu}}(\alpha^\nu)$.

<u>Corollary 4.1</u>

Let $\alpha^\nu \in A$ be given as in (4.4) and assume that (4.5) holds. Then for any $\Omega_{\bar{\nu}}$ such that $\bar{\nu} \geq \nu$ and $\Omega_\nu \subset \Omega_{\bar{\nu}}$, we have

$$\Psi_{\nu}(\alpha^{\nu}) \leq \Psi_{\bar{\nu}}(\alpha^{\nu}) \leq \Psi(\alpha^{\nu}) \leq \gamma_{\bar{\nu}}\, \Psi_{\bar{\nu}}(\alpha^{\nu}) + \omega_{\bar{\nu}} \tag{4.9}$$

Proof: The first two inequalities follow directly from $\Omega_{\nu} \subset \Omega_{\bar{\nu}} \subset \Omega$, while the third is given by (4.5) with $\alpha = \alpha^{\nu}$. ∎

5. PRODUCT SPLINE BASIS

In order to apply the previous results it is necessary to use an appropriate basis for the tensor product of splines of specified degree m with specified knots. A suitable such basis for computation is obtained by a slight modification of B-splines [5]. For a specified degree m and uniform knot size, this modification uses a single function $\beta_m(\tau)$ and forms a basis on the interval $x \in [0,1]$ by a linear combination of functions $\beta_m(\mu x - i)$. This representation, with the properties discussed below, simplifies the computation by leading to well-conditioned matrices with a special structure. For simplicity we discuss only splines of odd degree; a similar basis can be used for splines of even degree.

We define for $m = 2k - 1$,

$$\beta_m(\tau) = \frac{1}{m!} \sum_{j=-k}^{k} (-1)^{j+k} \binom{2k}{j+k} (j-\tau)_+^m , \tag{5.1}$$

where

$$(x)_+^m = \begin{cases} x^m, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{5.2}$$

The function $\beta_m(\tau)$ is symmetric about $\tau = 0$, bell-shaped and non-negative on the interval $[-k,k]$, and vanishes identically outside this interval. More specifically, it has the following properties:

$$\beta_m(\tau) \begin{cases} > 0, & |\tau| < k \\ = 0, & |\tau| = k \\ \equiv 0, & |\tau| > k \end{cases}$$

(5.3)
$$\beta_m(-\tau) = \beta_m(\tau)$$

$$\beta_m(0) > \beta_m(\tau), \ \tau \neq 0$$

$$\sum_{i=-\infty}^{\infty} |\beta_m(i)| = \sum_{i=1-k}^{k-1} \beta_m(i) = 1$$

The derivatives $\beta_m^{(\ell)}(\tau)$, $\ell = 1,2,\ldots,m$, of $\beta_m(\tau)$, are given by

(5.4)
$$\beta_m^{(\ell)}(\tau) = \frac{1}{(m-\ell)!} \sum_{j=-k}^{k} (-1)^{j+k+\ell} \binom{2k}{j+k} (j-\tau)_+^{m-\ell}$$

It follows that

$$\beta_m(\tau) \in C^{m-1}$$

(5.5)
$$\beta_m^{(\ell)}(\pm k) = 0, \ \ell = 0,1,\ldots,m-1$$

$$\beta_m^{(m+1)}(\tau) = 0, \text{ for noninteger } \tau$$

Furthermore, since $(x)_+^0 = 1$ for $x > 0$, and $(x)_+^0 = 0$ for $x \leq 0$, we also have that $\beta_m^{(m)}(\tau)$ is piecewise constant with discontinuities at $\tau = -k,\ldots k$, and $\beta_m^{(m)}(\tau) = \beta_m^{(m)}(j)$ for $j \leq \tau < j+1$.

Now consider the interval $[0,1]$ and a uniform knot size Δ, with $\mu\Delta = 1$. It can be shown that the $n = \mu + m$ functions $\beta_m(\mu x - i)$, $i = 1-k, \ldots \mu+k-1$, are linearly independent on $[0,1]$. Therefore an arbitrary spline of degree m on $[0,1]$, with $\mu + 1$ knots at $x = i\Delta$, $i = 0,1,\ldots\mu$, can be represented by

$$(5.6) \qquad s_m(\alpha, x) = \sum_{i=1-k}^{\mu+k-1} \alpha_i \beta_m(\mu x - i)$$

Because of the compact support for each $\beta_m(\mu x - i)$ there are at most $m+1$ nonzero terms in this summation for any fixed $x \in [0,1]$. Furthermore, it can be shown that for any $x \in [0,1]$,

$$(5.7) \qquad \sum_{i=1-k}^{\mu+k-1} \beta_m(\mu x - i) = 1$$

so that we always have the bound

$$(5.8) \qquad \|s_m\| \equiv \sup_{x\in[0,1]} |s_m(\alpha,x)| \le \max_i |\alpha_i|$$

Finally the derivatives of $s_m(\alpha, x)$ are given by the easily computed expressions

$$(5.9) \qquad \frac{d^\ell}{dx^\ell} s_m(\alpha,x) = \mu^\ell \sum_{i=1-k}^{\mu+k-1} \alpha_i \beta_m^{(\ell)}(\mu x - i)$$

where the $\beta_m^{(\ell)}(\tau)$ are given by (5.4).

Now consider a rectangular domain $\Omega \subset E^{\ell}$, as given by (3.13). Let $S_m(\Delta_j)$ denote the class of tensor product splines of degree m on Ω, with uniform knot size Δ_j corresponding to the variable x_j as given by (3.14). A basis for $S_m(\Delta_j)$ can readily be constructed using the single function $\beta_m(\tau)$. To accomplish this we represent any spline in $S_m(\Delta_j)$ by products of the functions $\beta_m(\frac{x_j}{\Delta_j} - i_j)$. Specifically from (3.14) and (5.6) we obtain

$$(5.10) \qquad S_m(\alpha, x) = \sum_{\substack{i_j = 1-k \\ j=1,\ldots \ell}}^{\mu_j + k - 1} \alpha_{i_1, i_2, \ldots i_\ell} \prod_{j=1}^{\ell} \beta_m(\frac{x_j}{\Delta_j} - i_j)$$

where the coefficient vector $\alpha \in E^n$, and $n = \prod_{j=1}^{\ell} (\mu_j + m)$. Because of the compact support, there are at most $(m+1)^{\ell}$ nonzero terms in this summation for any fixed $x \in \Omega$. Furthermore, using (5.8) and an induction on j, it can be shown that

$$(5.11) \qquad \|S_m\| = \sup_{x \in \Omega} |S_m(\alpha, x)| \leq \max_{i_j} |\alpha_{i_1, i_2, \ldots i_\ell}|$$

Other properties of $S_m(\alpha, x)$ follow directly from the properties of $\beta_m(\tau)$. In particular we have $D_{x_j}^{m+1} S_m(\alpha, x) = 0$ for x_j/Δ_j non-integer.

To illustrate the behavior of this representation of product splines a contour map of $\beta_3(x)\ \beta_3(y)$ is shown in Fig. 1. The square $x \in [-2,2]$ $y \in [-2,2]$ of compact support is shown with contour lines given by

$$\beta_3(x)\ \beta_3(y) = .05\ [\beta_3(0)]^2 i,\ i = 0,1,\ldots\ 19. \tag{5.12}$$

The contour for $i = 0$ is the boundary of the square. It is rather surprising to observe that all other contours shown are almost circular.

The representation (5.10) of a product spline is of the form (4.1), so that Theorem 3 can now be applied directly to show convergence of the product spline approximation as $h \to 0$.

<u>Theorem 4</u>

Let the assumptions of Corollary 3.2 hold. Then (4.6) holds with $\psi(\alpha) = \|f - S_m(\alpha)\|$, $\gamma_\nu = \lambda_m^\ell(h,\frac{1}{2})$ and

$$\omega_\nu = \bar{\lambda}_m(h,\tfrac{1}{2})h^{m+1} \sum_{k=0}^{\ell-1} \sigma_{\ell-k}\ \lambda_m^k\ (h,\tfrac{1}{2}).$$

<u>Proof</u>: The assumption (4.5), with γ_ν and ω_ν as given, is satisfied for $S_m(\alpha,x)$ as shown by Corollary (3.2). Therefore Theorem 3 holds with $V(\alpha,x) = S_m(\alpha,x)$. ▮

6. LINEAR PROGRAMMING FORMULATION AND COMPUTATIONAL RESULTS

As shown in Section 3, we obtain a minimum error bound by minimizing the error over the discrete grid Ω_h. We will now summarize the linear programming formulation which finds this minimum over Ω_h. This formulation will be given in the context of approximation by a generalized polynomial $V(\alpha,x)$ as given by (4.1) on a discrete grid $\Omega_\nu \subset \Omega$ of ν points. For the purposes of this formulation these ν points may be selected in any convenient manner.

We assume that we are given a function $f(x)$ on $\Omega \subset E^\ell$ and that a best approximation $V(\alpha^*,x)$ to $f(x)$ exists on Ω. We also assume we know a bounded polyhedral set $A \subset E^n$, such that $\alpha^* \in A$. We introduce a scalar variable ξ and consider the following problem

$$(6.1) \qquad \min_{\xi,\alpha} \left\{ \xi \;\middle|\; \begin{matrix} -\xi \le V(\alpha,x) - f(x) \le \xi, \quad \forall\, x \in \Omega_\nu \\ \alpha \in A \end{matrix} \right\}$$

This is a linear programming problem with $n+1$ variables and $2\nu + \bar{n}$ inequality constraints, where $\bar{n} \ge n+1$ is the number of inequality constraints required to define A. The optimal solution to (6.1), say ξ^{ν} and $\alpha^{\nu} \in A$ attain the minimum error over Ω_{ν}. That is,

$$(6.2) \qquad \psi_{\nu}(\alpha^{\nu}) = \|V(\alpha^{\nu}) - f\|_{\nu} = \xi^{\nu}$$

If is computationally more efficient to treat (6.1) as an unsymmetric dual problem and solve the equivalent primal problem using the standard simplex method [6]. This leads to a primal problem with $n + 1$ rows and $2\nu + \bar{n}$ columns. The computing time therefore depends primarily on the number n of functions, and only in a secondary way on the number ν of grid points. The values of f on Ω_{ν} appear as the "cost row" in the primal problem, making it easy to use multiple cost row or parametric features of most linear programming codes to solve a sequence of problems with different functions $f(x)$. In addition to the approximate solution $V(\alpha^{\nu}, x)$ and the maximum error ξ^{ν} on Ω_{ν}, the primal solution basis also gives a set of n+1 points in Ω_h at which the maximum error is attained. Details of this formulation are given in [4] and [8].

As an example, consider approximation by the product spline basis $S_m(\alpha, x)$ as given by (5.10) on a rectangular domain Ω, using a uniform grid Ω_{ν}. The grid Ω_{ν} is constructed with a uniform interval $h_j = \Delta_j/q$, corresponding to the coordinate x_j. We then have $n = \prod_{j=1}^{\ell} (\mu_j + m)$ and $\nu = \prod_{j=1}^{\ell} (q\mu_j + 1)$.

In order to illustrate the material discussed above, spline approximations have been obtained for a number of selected test problems. For each problem the approximation $s_m(\alpha^\nu, x)$ or $S_m(\alpha^\nu, x)$ was determined using linear programming. Error bounds were also computed using Corollary 2.1 on the unit interval and Corollary 3.1 on the half-square. The best available a priori error bounds for the same problems are also given for comparison purposes.

Results for two test problems on the unit interval $x \in I = [0,1]$ are presented in Table 1, the first with $f = e^{2x}$ and the second with $f = \sqrt{0.01 + x}$. A fixed knot size $\Delta = 0.1$ ($\mu = 10$) was used with both cubic and quintic splines and several different values for the grid size $h = \Delta/q$. No explicit restriction was placed on α, that is we took $A = E^n$. The representation $s_m(\alpha, x)$ given by (5.6) was used and the best approximation $s_m(\alpha^\nu, x)$ on the grid I_ν, with $h = (\nu-1)^{-1}$, was determined by linear programming as discussed above. The error $\psi_\nu(\alpha^\nu)$ on the grid I_ν and the bound on $\psi(\alpha^\nu) = \|f - s_m(\alpha^\nu)\|$ as given by (2.8) are tabulated for each case. For comparison the table also gives the a priori error bounds of

Hall [7] and Schultz [11] for the corresponding spline s_m determined by interpolation at the knots. A more accurate bound on $\psi(\alpha^\nu)$ is also obtained by the use of a finer grid and Corollary (4.1). This finer grid $I_{\bar{\nu}}$ with $\bar{\nu} > \nu$ points is chosen with $\bar{h} = (\bar{\nu} - 1)^{-1}$ sufficiently small so that

(6.3) $$\psi_{\bar{\nu}}(\alpha^\nu) \leq \psi(\alpha^\nu) \leq \hat{\lambda}_m \psi_{\bar{\nu}}(\alpha^\nu)$$

for some selected $\hat{\lambda}_m > \lambda_m$. In particular we choose

(6.4) $$\bar{h}^{m+1} \leq (\hat{\lambda}_m - \lambda_m) \psi (\alpha^\nu)/\bar{\lambda}_m \|f^{(m+1)}\|$$

The values $\hat{\lambda}_3 = 2.0$ and $\hat{\lambda}_5 = 4.0$, were selected. The value of $\psi_{\bar{\nu}}(\alpha^\nu)$ is tabulated and in general will be a good estimate for the error $\psi(\alpha^\nu)$, as well as a lower bound. Note that the determination of $\psi_{\bar{\nu}}(\alpha^\nu)$ requires no further minimization; only evaluation of the error over the finer grid $I_{\bar{\nu}}$. It should also be remarked that additional points may be added with no difficulty to the grid I_ν used in the minimization and to the finer grid $I_{\bar{\nu}}$ used for evaluation. These additional points should be added where $\|f^{(m+1)}\|$ is largest, and will improve both the approximation and the error bound. This was done for the test problems with $f = \sqrt{.01 + x}$ by adding 10 (unequally spaced) points in the interval [0, .05] for both I_ν and $I_{\bar{\nu}}$.

Results for the more difficult problem of approximation in two dimensions are presented in Table 2. The domain consists of the half-square $x \in [0,1]$, $y \in [0, 0.5]$, with the function $f = f(x,y) = e^{2xy^2}$. The product spline basis $S_m(\alpha,x,y)$ given by (5.10) was used with constant knot sizes Δ_x and Δ_y in the x and y directions. A uniform square grid Ω_ν, with $h = .0625$ and $\nu = 153$ was used in the linear programming minimization to obtain α^ν and $\psi_\nu(\alpha^\nu)$. Again no explicit restriction was placed on α. A finer uniform square grid $\Omega_{\bar{\nu}}$ with $\bar{h} = .02$ and $\bar{\nu} = 1326$ was used for the evaluation to get $\psi_{\bar{\nu}}(\alpha^\nu)$ and the error bound. The error bound was computed using (3.2) with $\ell = 2$ and the first relations in (3.3) and (3.4). For comparison, the interpolating spline approximation (interpolation on Ω_ν with $h_x = \Delta_x$, $h_y = \Delta_y$) and the corresponding a priori error bounds of Schultz [11] are also given.

It is of some interest to see the error curve along a diagonal line for this two-dimensional domain. This is shown in Fig. 2 for the cubic product spline approximation obtained with $h = .0625$, along the line $y = 0.5x$. Along coordinate directions the error curve is close to a Chebyshev polynomial as in the one-dimensional case. Along the diagonal line the error curve is seen to oscillate but with unequal positive and negative peaks.

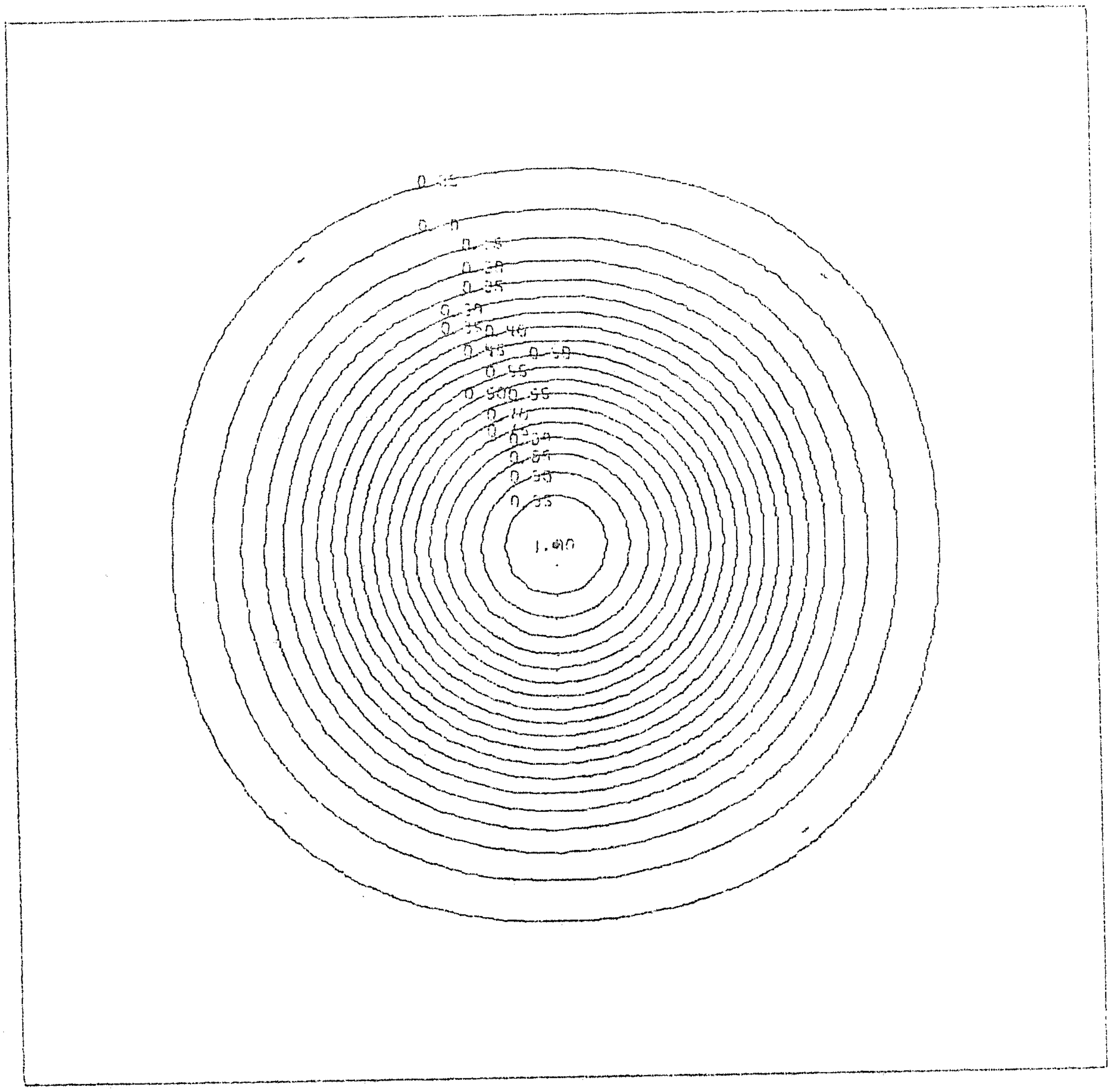

Figure 1

Contour plot for $\beta_3(x)\,\beta_3(y)$

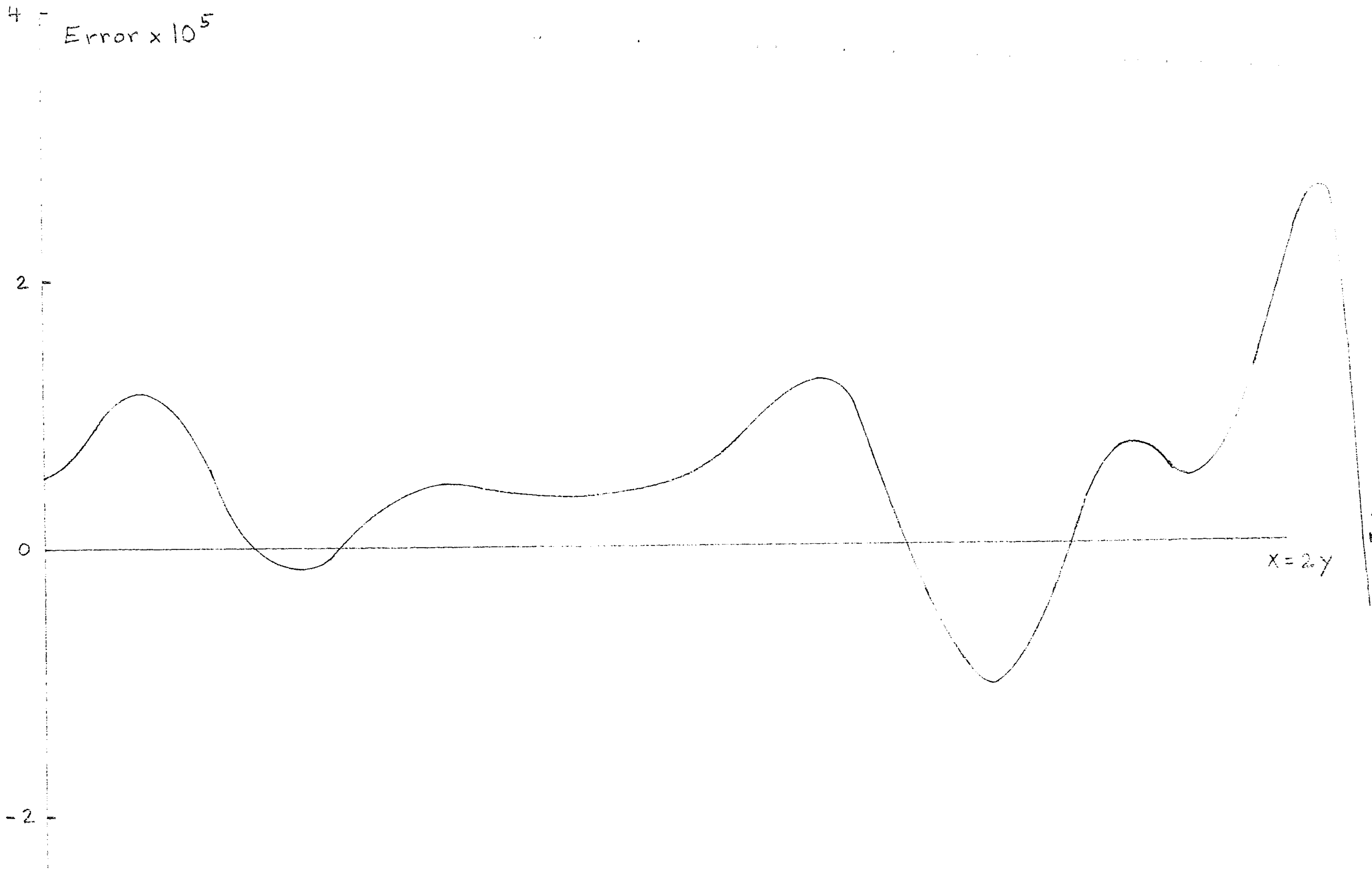

Figure 2

Approximation by product spline on half-square.
Error along diagonal line. Error = $e^{2xy^2} - S_m$

Table 1

Spline Approximation on Unit Interval

$x \in [0,1], \ \Delta = 0.1$

f	m	h	$\Psi_\nu(\alpha^\nu)$	$\Psi_{\bar{\nu}}(\alpha^\nu)$	Error Bounds Eqn. (2.8)	Hall	Schultz
e^{2x}	3	0.1	0	2.97E-5	--	1.5E-4	2.4E-3*
e^{2x}	3	0.025	1.15E-5	1.36E-5	2.8E-4	--	--
e^{2x}	3	0.0125	1.15E-5	1.20E-5	4.3E-5	--	--
e^{2x}	5	0.1	0	5.5E-8	--	3.1E-8	8.7E-5*
$\sqrt{.01+x}$	3	0.1	0	1.92E-2	--	12.2	0.36**
$\sqrt{.01+x}$	3	0.0125	2.31E-3	2.32E-3	8.2E-3	--	--
$\sqrt{.01+x}$	5	0.1	0	3.2E-2	--	96.5	3.6**
$\sqrt{.01+x}$	5	0.00625	6.6E-4	7.4E-4	3.6E-3	--	--

*Theorem 2.9, reference [11]

**Theorem 2.8, reference [11]

Table 2

Approximation by Product Spline on Half-Square

$x \in [0,1]$, $y \in [0,0.5]$, $f = e^{2xy^2}$, $\bar{h} = 0.02$

m	Δ_x	Δ_y	h	n	$\Psi_\nu(\alpha^\nu)$	$\Psi_{\bar{\nu}}(\alpha^\nu)$	Error Bounds Eqn. (3.2)	Error Bounds Schultz*
3	0.333	0.2	**	48	0	2.0E-4	7.6E-4	.024
3	0.333	0.2	.0625	48	9.37E-6	2.72E-5	2.6E-4	--
5	0.5	0.25	**	49	0	1.82E-5	2.1E-4	0.27
5	0.5	0.25	.0625	49	1.14E-5	1.7E-5	2.0E-4	--

*Theorem 3.1 of reference [11]

**Interpolating spline; $h_x = \Delta_x$, $h_y = \Delta_y$.

7. EXTENSION TO LINEAR BOUNDARY VALUE PROBLEMS

The previous results for the approximation of a given function by splines can be extended in a natural manner to the approximate solution of certain linear boundary value problems. Specifically we consider problems defined by a linear differential operator with constant (or polynomial) coefficients. For such problems the error bounds and computational method are applicable with a minimum of difficulty.

A closely related approach for both linear and nonlinear boundary value problems, which however does not take advantage of the nice properties of splines, is described in [10]. Other methods for the approximate solution of boundary value problems using splines have been presented, for example in [2] and [11].

We first consider a differential equation on the interval $I = [0,1]$. For a fixed $q \geq 1$, let

$$(7.1) \qquad L[u] \equiv \sum_{j=0}^{q} a_j(x)\, D^j u$$

where $a_j(x)$ is a polynomial of degree j on I, that is $a_j(x) \in \Pi_j$, and $D^j u = u^{(j)}$. Given a function $g \in C^{m+1}[I]$, with $m \geq q+1$, we consider the problem

$$(7.2) \qquad L[u] = g \quad \text{on} \quad I$$

subject to q specified linear boundary conditions at $x = 0$ (initial value) or at both $x = 0,1$ (two-point boundary value). We will denote these q boundary conditions by

(7.3) $\qquad B[u] = b$

It will be assumed that a unique solution $u(x)$ on I exists satisfying (7.2) and (7.3).

The solution is approximated by a spline $s_m(\Delta;x)$ with uniformly spaced knots at a set of points $I_\Delta \subset I$. We restrict the class of splines considered to those satisfying (7.3). Since $m \geq q+1$, $L[s_m]$ is continuous on I. Furthermore, since $s_m \in \Pi_m$ on $I - I_\Delta$, $D^j s_m \in \Pi_{m-j}$, so that $L[s_m] \in \Pi_m$ on $I - I_\Delta$. Therefore

(7.4) $\qquad D^{m+1} L[s_m] = 0 \quad \text{on} \quad I - I_\Delta$

The relevant error now is that in the differential equation (7.2). We therefore let

(7.5) $\qquad \rho(x) = g(x) - L[s_m]$

and now obtain

Theorem 5

Assume that I_ν is constructed as described for Corollary 2.1

[Corollary 2.2]. Then the error bound given by (2.8) [second inequality of (2.8)] holds with $g - L[s_m]$ replacing $f - P_m$ and $g^{(m+1)}$ replacing $f^{(m+1)}$.

Proof: The proof of Theorem 1 applies directly to this case using (7.5) and (7.4). The extension to the interval I is essentially the same as given in Corollaries 2.1 and 2.2. ∎

It should be pointed out that an important special case of this theorem occurs when $g^{(m+1)} = 0$. This immediately gives the bounds

$$(7.6) \qquad \|g - L[s_m]\| \le \begin{cases} \lambda_m \|g - L[s_m]\|_h , & h \le \Delta/m \\ [1 + \eta_m(h,1)h]\|g - L[s_m]\| , & h \le \Delta/m^2 \end{cases}$$

Theorem 5 gives us a bound on the uniform error in the differential equation $\|g - L[s_m]\|$ in terms of the corresponding error on I_v. In order to bound the error in the approximate solution $\|u - s_m\|$ we need an additional relation between these two quantities. Many problems of the form (7.2) and (7.3), with unique solutions, possess the following monotone property which gives us this additional relation.

Monotone Property: There exists a constant K such that if w is any function with $B[w] = b$ and $L[w]$ continuous on I, then

(7.7) $$\|u - w\| \leq K\|L[u] - L[w]\| .$$

Since $w = s_m$ satisfies these conditions and since $L[u] = g$, this monotone property immediately allows us to use Theorem 5 to bound $\|u - s_m\|$ in terms of the computed (minimized) quantity $\|g - L[s_m]\|_h$.

In order to obtain the approximate solution to (7.2) and (7.3) we again assume a uniform knot size and use the representation for s_m in terms of the function $\beta_m(\tau)$ as given by (5.6). Linear programming may again be used in essentially the same manner as discussed in Section 6.

First considering the boundary conditions (7.3), we use the linearity of B and (5.6) to give

(7.8) $$B[s_m] = \sum_i \alpha_i B[\beta_m] = b$$

These relations give q linear equations on the coefficients α_i . For example, in the initial value problem, $B[u] = b$ represents $u^{(j)}(0) = b_j$, $j = 0,1,\cdots q - 1$. Then (7.8) requires that the α_i satisfy

(7.9) $$\sum_{i=1-k}^{\mu+k-1} \alpha_i \beta_m^{(j)}(-i) = b_j/\mu^j, \quad j = 0,1,\cdots q - 1$$

These q linear equations are then included in the polyhedral set A. In order to minimize the error $\|g - L[s_m]\|_h$ on the discrete set I_ν, we let

$$(7.10) \qquad \hat{\beta}_i(x) = L[\beta_m(\mu x - i)]$$

The minimization is then carried out by solving (6.1) with

$$(7.11) \qquad V(\alpha, x) = \sum_{i=1-k}^{\mu+k-1} \alpha_i \hat{\beta}_i(x)$$

The solution to this linear programming problem gives the coefficient vector α^ν, and the corresponding function $s_m(\alpha^\nu, x)$, given by (5.6), which satisfies the boundary conditions and minimizes the error in the differential equation on the discrete set I_ν. Provided a monotone property holds, a bound on the error $\|u - s_m(\alpha^\nu)\|$ can be easily obtained using Theorem 5.

It should also be noted that the determination of the approximate solution by linear programming does not depend on the coefficients $a_j(x)$ being polynomials. The computational method described can be used to obtain an approximate solution assuming only that the coefficients are continuous on I. Of course, for this more general case the error bounds may not be valid.

In order to extend the results to linear partial differential equations we consider a closed and bounded rectangular domain

$\Omega \subset E^{\ell}$ as in Section 3. We denote by $\partial\Omega$ the boundary of Ω. We limit consideration to linear partial differential equations with constant coefficients. Let $L[u]$ denote a partial differential operator with constant coefficients. It is assumed that the highest partial derivative with respect to any one variable occuring in $L[u]$ is no greater than q. Given a function $g \in C^{m+1}[\Omega]$, with $m \geq q + 1$, we consider the problem

$$(7.12) \qquad L[u] = g \quad \text{on} \quad \Omega$$

subject to appropriate boundary conditions. The boundary conditions are assumed to be given in terms of one or more linear boundary operators with constant coefficients. If differential operators are involved they satisfy the same restriction on partial derivatives as $L[u]$. We denote these boundary conditions by

$$(7.13) \qquad B[u] = b \quad \text{on} \quad \partial\Omega$$

For example B might be the identity operator or the normal derivative on each boundary face. As in the one-dimensional case we assume that a unique solution $u(x)$ on Ω exists satisfying (7.12) and (7.13).

For this multidimensional problem the solution is conveniently approximated by a product spline $S_m(\Delta;x)$ as used in Corollary 3.1

or Corollary 3.2. Because of the derivative restriction assumed on L and B, we have that $L[S_m]$ and $B[S_m]$ are continuous on Ω and $\partial\Omega$ respectively. Furthermore, since $L[S_m]$ and $B[S_m]$ are at most polynomials of degree m in x_i on $\Omega - \Omega_\Delta$, we have $D_{x_i}^{m+1} L[S_m] = 0$ and $D_{x_i}^{m+1} B[S_m] = 0$ on $\Omega - \Omega_\Delta$.

We could apply the results of Sections 3 and 5 directly to this multidimensional problem as we did in the one-dimensional case, except for the one significant difference that, in general, we can no longer satisfy the boundary conditions exactly. We must therefore take into account the error in both the differential equation and the boundary conditions. To apply the bounds of Corollaries 3.1 and 3.2 to the error in the differential equations we let $\rho = g - L[S_m]$ on Ω and assume that $\| D_{x_j}^{m+1} g \|_\Omega \le \sigma_j$, $j = 1, \ldots \ell$. Similarly for the boundary error we let $\rho = b - B[S_m]$ on $\partial\Omega$, and assume that $\| D_{x_j}^{m+1} b \|_{\partial\Omega} \le \sigma_j$, $j = 1, \ldots \ell$. We then obtain

Theorem 6

Assume that the discrete grid $\Omega_\nu \subset \Omega$ and $\partial\Omega_\nu \subset \partial\Omega$, is constructed as for Corollary 3.1 or Corollary 3.2. Then the error bound (3.2) holds on Ω with $g - L[S_m]$ replacing $f - P_m$, and on $\partial\Omega$ with $b - B[S_m]$ replacing $f - P_m$. ▮

In order to make the best use of these error bounds in determining an approximate solution we again need a monotone property to relate the errors in (7.12) and (7.13) to the error $\|u - S_m\|_\Omega$. For many boundary value problems a monotone property can be obtained in the following form.

Monotone Property: There exist constants K_d and K_b such that if w is any function with $L[w]$ and $B[w]$ continuous on Ω and $\partial\Omega$ respectively, then

(7.14) $$\|u - w\|_\Omega \le K_d \|L[u] - L[w]\|_\Omega + K_b \|B[u] - B[w]\|_{\partial\Omega}$$

For computational purposes the product spline representation $S_m(\alpha;x)$ given by (5.10) is again used. Assuming the monotone property and in view of Theorem 6, we can minimize the error bound by solving a linear programming problem related to (6.1). To illustrate, we consider the special case where B is the identity operator so that (7.13) requires $u = b$ on each $\ell - 1$ dimensional face of the rectangular domain Ω. Thus the boundary conditions, if considered alone, would lead to a problem in $E^{\ell-1}$ essentially as given by (6.1) with $f = b$, for each face of Ω. We will also assume that $D_{x_i}^{m+1} g = D_{x_i}^{m+1} b = 0$, $i = 1, \ldots \ell$. Choose any discrete grid $\Omega_\nu \subset \Omega$ and $\partial\Omega_\nu \subset \partial\Omega$ satisfying the requirements of Corollary 3.1.

Then from Theorem 6 and (7.14) we have

$$\|u - S_m\|_\Omega \le K_d \lambda_m^\ell \|L[S_m] - g\|_{\Omega_\nu} + K_b \lambda_m^{\ell-1} \|S_m - b\|_{\partial\Omega_\nu} \tag{7.15}$$

Thus to find $S_m(\alpha, x)$ so as to minimize the error bound we introduce two scalar variables ξ and ζ, and solve the linear programming problem

$$\min_{\xi,\zeta,\alpha} \left\{ K_d \lambda_m^\ell \xi + K_b \lambda_m^{\ell-1} \zeta \;\middle|\; \begin{array}{l} -\zeta \le L[S_m(\alpha,x)] - g(x) \le \xi \;\forall\, x \in \Omega_\nu \\ -\zeta \le S_m(\alpha,x) - b(x) \le \zeta \;\forall\, x \in \partial\Omega_\nu \end{array} \right\} \tag{7.16}$$

The optimal solution gives the values $\hat{\xi}$, $\hat{\zeta}$, α^ν, the approximating spline $S_m(\alpha^\nu, x)$, and the minimum error bound $K_d \lambda_m^\ell \hat{\xi} + K_b \lambda_m^{\ell-1} \hat{\zeta}$ for $\|u - S_m(\alpha^\nu)\|_\Omega$.

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