FINITE DIFFERENCE METHODS AND THE
EIGENVALUE PROBLEM FOR NON
SELF-ADJOINT STURM-LIOUVILLE OPERATORS

by
Alfred Carasso

Technical Report # 46

October 1968
FINITE DIFFERENCE METHODS AND THE EIGENVALUE PROBLEM FOR NON SELF-ADJOINT STURM-LIOUVILLE OPERATORS*

by

Alfred Carasso**

1. INTRODUCTION

Many authors (e.g. [1], [6], [8], [9]) have studied the convergence of finite difference methods for self-adjoint Sturm-Liouville eigenvalue problems. In this report we are concerned with the non self-adjoint problem

\[(1,1) \quad \mathcal{L}(u) \equiv -[a(x)u']' - b(x)u' + c(x)u = \lambda u, \quad 0 < x < 1 \]

\[u(0) = u(1) = 0\]

where \(a(x) \geq a_0 > 0\), \(c(x) \geq 0\), and \(b(x)\) are all smooth functions. This problem has an infinite sequence of positive and distinct eigenvalues

\[0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots\]

and a corresponding sequence of smooth eigenfunctions \(u^1(x), u^2(x), u^3(x), \ldots\) which we assume normalized so that

\[(1,2) \quad \int_0^1 |u^p|^2 \, dx = 1, \quad p = 1, 2, \ldots\]

Of course, as is well known, the transformation

\[(1,3) \quad u(x) = \left[ \exp \left( -\frac{1}{2} \int_0^x \frac{b(t)}{a(t)} \, dt \right) \right] v(x)\]

*Supported by the Office of Naval Research under Contract Number N00014-67-A-1028-0004 at the University of Wisconsin. Reproduction in whole or in part is permitted for any purpose of the United States Government.

**Department of Mathematics, Michigan State University, East Lansing, Michigan.
puts (1.1) into the self adjoint form

\( \hat{\mathcal{L}} [v] \equiv -(av')' + (c + \frac{1}{2} b' + \frac{1}{4} \frac{b^2}{a})v = \lambda v \)

(1.4)

\( v(0) = v(1) = 0 \)

However, we consider the direct approximation of (1.1) by means of the finite difference equations

\[\begin{align*}
\frac{1}{\Delta x^2} & \left( a_{k+\frac{1}{2}} (w_{k+1} - w_k) - a_{k-\frac{1}{2}} (w_k - w_{k-1}) \right) - \frac{b_k (w_{k+1} - w_{k-1})}{2\Delta x} \\
& + c_k w_k = \Lambda w_k & k = 1, 2, \cdots M
\end{align*}\]

(1.5)

\( w_0 = w_{M+1} = 0 \)

where \( M \) is a large positive integer, \( \Delta x = \frac{1}{M+1} \) is the mesh spacing and the notation \( g_k \) is used for \( g(k\Delta x) \). Equivalently, we may write (1.5) as the finite dimensional eigenvalue problem:

(1.6) \( \mathbf{LW} = \Lambda \mathbf{W} \)

where \( \mathbf{W} \) is the \( M \) component vector \( \mathbf{W} = [w_1 \ w_2 \ \cdots \ w_M]^T \) and \( \mathbf{L} \) the \( M \times M \) tridiagonal matrix

(1.7) \( \mathbf{L} = \frac{1}{\Delta x^2} \)

\[\begin{array}{cccccccc}
\alpha_1 & \beta_1 & \gamma_2 & \alpha_2 & \beta_2 & \cdots & \cdots & \beta_{M-1} \\
\gamma_2 & \alpha_2 & \beta_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_M & \alpha_M & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\]
Lemma 1

There exists a non-singular, positive, diagonal matrix $D$ such that $D^{-1} L D = \hat{L}$ is a real symmetric matrix. Moreover, $\|D\|_2, \|D^{-1}\|_2$ remain bounded as $M \to \infty$, $\Delta x \to 0$, $(M + l) \Delta x = 1$.

Proof: We construct such a matrix.

Let $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & \cdots \end{bmatrix}$ where $d_j \neq 0$, $j = 1, \ldots, M$ and $d_1 = 1$.

Let $\hat{L} = D^{-1} L D = (\hat{L}_{ij})$.

Since we require $\hat{L} = \hat{L}^T$, we must have

$$d_i^{-1} \hat{L}_{ij} d_j = d_j^{-1} \hat{L}_{ji} d_i$$

where $L = (\hat{L}_{ij})$

Further, since $\hat{L}_{ij} = 0$ for $j > i + 1$, $j < i - 1$, the $d_j$'s must be determined so that

$$d_i^2 = \frac{\hat{L}_{i, i-1}}{\hat{L}_{i-1, i}} \quad d_1^2 = 1, \ldots, M$$

Starting from $d_1 = 1$, we may solve recursively to obtain

$$d_i^2 = \prod_{k=1}^{i-1} \left( \frac{\gamma_k + 1}{\beta_k} \right) \quad i = 2, \ldots, M$$

and, since $\gamma_k, \beta_k < 0$ for sufficiently small $\Delta x$, $d_i^2 > 0$ if $\Delta x$ is small enough.
With $D$ constructed as above, we have

\[
\hat{L} = \frac{1}{\Delta x^2} \begin{bmatrix}
\alpha & -\gamma_2 \beta_1^{\frac{1}{2}} \\
\vdots & \ddots & \ddots & \ddots \\
-\gamma_2 \beta_1^{\frac{1}{2}} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & -\gamma_{M-2} \beta_{M-2}^{\frac{1}{2}} \\
\gamma_{M-2} \beta_{M-2}^{\frac{1}{2}} \cdot a_M & -\gamma_{M-1} \beta_{M-1}^{\frac{1}{2}} \cdot a_M & \cdots & \cdots & \cdots & 0
\end{bmatrix}
\]

and we must show that $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $M \to \infty$.

Let $Q_i = \prod_{k=1}^{i-1} \left(1 - \frac{b_{k+1}}{2a_k} \Delta x \right)$ and $P_i = \prod_{k=1}^{i-1} \left(1 + \frac{b_k}{2a_k} \Delta x \right)$
then $d_i^2 = \frac{Q_i}{P_i}$. Now for sufficiently small $\Delta x$,

\[
\log \left(1 - \frac{b_{k+1} \Delta x}{2a_k} \right) = -\frac{b_{k+1} \Delta x}{2a_k} + O(\Delta x^2)
\]
so that

\[
\log Q_i = -\Delta x \sum_{k=1}^{i-1} \frac{b_k}{2a_k} + \Delta x \sum_{k=1}^{i-1} O(\Delta x)
\]

Hence,

\[
\lim_{\Delta x \to 0, \ i \to \infty} \left[ \log Q_i \right] = -\frac{1}{2} \int_0^{\tilde{X}} \frac{b(t)}{a(t)} \, dt
\]
Similarly,

\[
\lim_{\Delta x \to 0, \; i \to \infty} \left[ \log Q_1 \right] = \frac{1}{2} \int_0^\infty b(t) \frac{1}{a(t)} \, dt
\]

Consequently, \( \lim d_i = e^{-\frac{1}{2} \int_0^\infty b(t) \frac{1}{a(t)} \, dt} \leq K_0 < \infty \) which shows both \( \|D\|_2, \; \|D^{-1}\|_2 \) remain bounded as \( \Delta x \to 0, \; M \to \infty, \; (M+1)\Delta x = 1 \).

**Lemma 2**

For \( \Delta x \) sufficiently small, the eigenvalues of \( L \) are strictly positive and they remain bounded away from zero as \( M \to \infty, \; \Delta x \to 0, \; (M+1)\Delta x = 1 \).

**Proof:** For \( \Delta x \) sufficiently small, \( \gamma_k, \; \beta_k < 0 \). Hence if \( L = (\ell_{ij}) \) and

\[
\Omega_i = \sum_{j \neq i} |\ell_{ij}|, \quad \text{then}
\]

\[
\Omega_i = \frac{a_{i+\frac{1}{2}} + a_{i-\frac{1}{2}}}{\Delta x^2}
\]

and

\[
\ell_{ii} = \frac{a_{i+\frac{1}{2}} + a_{i-\frac{1}{2}}}{\Delta x^2} + c_i \geq \Omega_i \text{ since } c_i \geq 0.
\]

By Gershgorin's theorem, ([7]), the eigenvalues of \( L \) lie in the union of the discs

\[
|z - \ell_{ii}| \leq \Omega_i
\]

in the complex plane. Hence if \( \Lambda \) is an eigenvalue of \( L \), then \( \Lambda \geq 0 \) since \( \Lambda \) is real.

Now let \( \ell_h \) be the finite difference operator corresponding to \( -L \) i.e.
\[ [\mathbf{L}_h \mathbf{v}]_k \equiv \begin{bmatrix} (a_{k+\frac{1}{2}} + a_{k-\frac{1}{2}} + c_k \Delta x^2) \\ \frac{a_{k+\frac{1}{2}} + a_{k-\frac{1}{2}}}{\Delta x^2} \\ \frac{a_{k-\frac{1}{2}} - b_k \Delta x}{\Delta x^2} \end{bmatrix} \mathbf{v}_k + \begin{bmatrix} a_{k+\frac{1}{2}} + b_k \Delta x \\ \frac{a_{k+\frac{1}{2}} + b_k \Delta x}{\Delta x^2} \\ \frac{a_{k-\frac{1}{2}} - b_k \Delta x}{\Delta x^2} \end{bmatrix} \mathbf{v}_{k+1} + \begin{bmatrix} \mathbf{v}_k \\ \mathbf{v}_k \\ \mathbf{v}_{k-1} \end{bmatrix} \]

Then, for sufficiently small \( \Delta x \), \( \mathbf{L}_h \) is of positive type and so satisfies the discrete maximum principle (See [3]). Consequently if \( w(k \Delta x) \), \( k = 0, 1, \ldots, M+1 \) is an arbitrary real valued mesh function, there exists positive constraints \( K \) and \( \delta \) such that if \( 0 < \Delta x < \delta \),

\[(2.2) \quad \|w\|_{\infty} \equiv \max_k |w_k| \leq \max \{|w_0|, |w_{M+1}|\} + K \|L_h w\|_{\infty} \]

Now let \( \mathbf{V} = \{\mathbf{v}_k\}_{k=1}^M \) be an eigenvector of \( \mathbf{L} \) corresponding to \( \Lambda \). We may assume \( \mathbf{V} \) to be real. Defining \( \mathbf{v}_0 = \mathbf{v}_{M+1} = 0 \), \( \mathbf{L} \mathbf{V} = \Lambda \mathbf{V} \) is equivalent to

\[(2.3) \quad [L_h \mathbf{v}]_k = -\Lambda \mathbf{v}_k \quad k = 1, \ldots, M.\]

Hence, using (2.2) and the fact that \( \Lambda \geq 0 \),

\[
\|\mathbf{v}\|_{\infty} \leq K \|L_h \mathbf{v}\|_{\infty} = \Lambda K \|\mathbf{v}\|_{\infty}
\]

i.e. \( \Lambda \geq \frac{1}{K} > 0 \) Q.E.D.
Corollary

Let \( \Gamma \) be the \( M \times M \) matrix given by

\[
\begin{bmatrix}
+1 & & & \\
& -1 & & \\
& & \ddots & \\
& & & (-1)^{M-1}
\end{bmatrix}
\]

then \( \Gamma^{-1} \tilde{L} \Gamma \) is an oscillation matrix.

Proof: \( \Gamma^{-1} \tilde{L} \Gamma \) is a positive definite real symmetric matrix with positive elements along the first super and sub diagonals. The proof now follows from a theorem of Gantmacher and Krein [4, p. 103].

3. CONVERGENCE OF THE CHARACTERISTIC PAIRS OF \( L \)

Let \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_M \) be the eigenvalues of \( L \). Fix a positive integer \( p \) and let \( V^p(\Delta x) \) be the eigenvector corresponding \( \lambda_p(\Delta x) \), normalized so that \( \| V^p \|_2 = 1 \). Let \( \tilde{V}^p \) be the continuous piecewise linear function, vanishing at \( x = 0, 1 \), and which, in the interior of \([0,1]\), is obtained from \( V^p \) by linear interpolation. Consider the families \( \{ \lambda_p(\Delta x) \} \), \( \{ \tilde{V}^p(\Delta x) \} \) as the mesh size \( \Delta x \to 0 \).

A direct proof of convergence of \( \tilde{V}^p \) to \( u^p(x) \) and \( \lambda_p \) to \( \lambda_p \) may be given, which is based on the compactness of the family \( \{ \tilde{V}^p(\Delta x) \} \) in \( C[0,1] \). Such an approach was used by PARTER in [9], (See also [2]); but
this method does not yield estimates on the rates of convergence.

Nevertheless we will make use of the fact that \( \Lambda_p \to \lambda_p \) together with lemma 1 above to obtain these estimates. The argument given below is a modification of that given by GARY in [6] for the self-adjoint case.

**Theorem 1**

Let \( \Lambda_p \), \( V_p \) be characteristic pairs of \( L \) with \( \| V_p \|_2 = 1 \). Let \( D \) be the diagonal matrix of Lemma 1. Let \( u_p \) be an eigenfunction of \( L \) corresponding to \( \lambda_p \) and let \( U_p \) be the \( M \) vector obtained from \( u_p \) by mesh-point evaluation. Assume \( u_p(x) \) normalized so that

\[
(3.1) \quad \| D^{-1} u_p \|_2 = \| D^{-1} V_p \|_2
\]

then as \( \Delta x \to 0 \), we have

\[
(3.2) \quad |\lambda_p - \Lambda_p| \leq K \Delta x^2
\]

\[
(3.3) \quad \| U_p - V_p \|_2 \leq K_1 \Delta x^2
\]

where \( K, K_1 \) are positive constants defining only \( a_p \).

**Proof:** Because the difference scheme in (1.5) is properly centered and we assume sufficient smoothness of \( u_p \) and the coefficients of \( L \), we have at the mesh points,

\[
(3.4) \quad L [u_p] = L u_p + \tau = \lambda_p u_p
\]

where \( \tau \) is the "truncation" error and
\[(3.5) \quad \| \tau \|_2^2 \leq K(p) \Delta x^2 \quad \text{where} \quad K \text{ is a constant.} \]

Let \( \hat{L} = D^{-1}L \) have orthonormal eigenvectors \( X^1, X^2, \ldots, X^M \) and write \( U^p \) as a linear combination of the \( DX^j \)'s:

\[(3.6) \quad U^p = \sum_{j=1}^{M} \sigma_j DX^j \]

so that

\[LU^p = \sum_{j=1}^{M} \sigma_j LDX^j = \sum_{j=1}^{M} \sigma_j \Lambda_j DX^j \]

then,

\[\tau = (\lambda_p - L)U^p = \sum_{j=1}^{M} \sigma_j (\lambda_p - \Lambda_j) DX^j \]

and

\[(3.7) \quad \sum_{j=1}^{M} \sigma_j^2 |\lambda_p - \Lambda_j|^2 = \| D^{-1} \tau \|_2^2 \leq \| D^{-1} \|_2^2 \| \tau \|_2^2 \leq K_1(p) \Delta x^4 \quad \text{where} \quad K_1 \]

is a constant.

Now, the eigenvalues of \( L \) are distinct and converge to the corresponding distinct eigenvalues of \( \Sigma \). It follows that

\[(3.8) \quad \inf_{j \neq p} \{ |\lambda_p - \Lambda_j| \} \geq \omega_0 > 0 \quad \text{for all sufficiently small} \quad \Delta x.\]

Hence, on using (3.7),

\[(3.9) \quad \sum_{j \neq p} \sigma_j^2 \leq K_1 \Delta x^4 \]
and

\[(3.10) \quad \sigma_p^2 = \| D^{-1} U^p \|_2^2 + O(\Delta x^4) \geq \omega_1 > 0 \]

for all sufficiently small \( \Delta x \).

Thus

\[(3.11) \quad |\lambda_p - \Lambda_p| \leq K_2(p) \Delta x^2 \]

Since \( V^p = \beta DX^p \) for some \( \beta \) and \( \| X^p \|_2 = 1 \) we have

\[|\beta| = \| D^{-1} V^p \|_2 \]

On taking square roots in (3.10), we have

\[\sigma_p = \| D^{-1} U^p \|_2 + O(\Delta x^4) \]

and we may assume that \( \sigma_p \) and \( \beta \) have the same sign; hence using (3.1),

\[(3.12) \quad (\sigma_p - \beta) = O(\Delta x^4) \]

Writing \( U^p - V^p = \sum_{j \neq p} \sigma_j DX^j + (\sigma_p - \beta) DX^p \)

we have

\[(3.13) \quad \| D^{-1} (U^p - V^p) \|_2^2 = \sum_{j \neq p} \sigma_j^2 + (\sigma_p - \beta)^2 = O(\Delta x^4) \]

i.e.

\[(3.14) \quad \| U^p - V^p \|_2^2 \leq \| D \|_2^2 \| D^{-1} (U^p - V^p) \|_2^2 \leq K_3(p) \Delta x^4 \quad Q.E.D. \]

Notice that the above inequality also implies uniform convergence at the rate of \( O(\Delta x)^{3/2} \).
4. PROOF OF THEOREM 2

Lemma 3

Let \( 0 < \Lambda_1 < \cdots < \Lambda_M \) be the eigenvalues of \( L \). Then there exists a positive integer \( j_0 \), independent of \( M \), such that for \( j_0 \leq j \leq M \) we have

\[
K_1 j^2 \pi^2 \leq \Lambda_j \leq K_2 j^2 \pi^2
\]

\( K_1, K_2 \) positive constants.

Proof: In the self adjoint case, this result may be found in Bückner [1]. In the present more general case see [2].

Proof of Theorem 2.

Let \( W_j = \begin{bmatrix} w_{1j} \\ \vdots \\ w_{Mj} \end{bmatrix} \) be an eigenvector of \( L \) corresponding to \( \Lambda_j \). Then \( W_j \) satisfies the difference equations.

\[
\begin{cases}
-2 + \left( \frac{c_k - \Lambda_j}{\omega_k} \Delta x \right)^2 & w_{kj}^j + \left( \frac{a_{k+\frac{1}{2}} + \frac{b_k \Delta x}{2}}{\omega_k} \right) w_{k+1j}^j \\
+ \left( \frac{a_{k-\frac{1}{2}} - \frac{b_k \Delta x}{2}}{\omega_k} \right) w_{kj-1}^j = 0 & k = 1, \ldots, M
\end{cases}
\]

\( w_0^j = w^j_{M+1} = 0 \) and \( \omega_k = \frac{1}{2} \left( a_{k+\frac{1}{2}} + a_{k-\frac{1}{2}} \right) \)
Let \( \tilde{\alpha}_k = \left[ 2 + \frac{(c_k - \Lambda_1) \Delta x^2}{\omega_k} \right] \tilde{\beta}_k = \frac{a_{k+\frac{1}{2}} + \frac{1}{3} b_k \Delta x}{\omega_k} \)

\[
\tilde{\gamma}_k = \frac{a_{k-\frac{1}{2}} - \frac{1}{3} b_k \Delta x}{\omega_k}
\]

and let \( A \) be the tridiagonal \( M \times M \) matrix

\[
(4.3) \quad A = \\
\begin{bmatrix}
\tilde{\alpha}_1 & \tilde{\beta}_1 & & & \\
\tilde{\gamma}_2 & \tilde{\alpha}_2 & \tilde{\beta}_2 & & \\
& \ddots & \ddots & \ddots & \\
& & \tilde{\gamma}_M & \tilde{\alpha}_M & \tilde{\beta}_{M-1} \\
& & & \tilde{\gamma}_M & \\
& & & & \tilde{\alpha}_M
\end{bmatrix}
\]

Then we may write (4.2) as

\[
(4.4) \quad AW^j = 0 \quad \text{or equivalently}
\]

\[
(4.5) \quad (P^{-1}AP) P^{-1}W^j = 0 \quad \text{if} \ P \ \text{is any non singular matrix.}
\]

Choose \( P \) to be the diagonal matrix

\[
(4.6) \quad P = \\
\begin{bmatrix}
p_1 & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & p_M
\end{bmatrix}
\]

where \( p_1 = 1 \) and \( p_i^2 = \prod_{k=1}^{i-1} \frac{\tilde{\gamma}_k + 1}{\tilde{\beta}_k} \quad i = 2, \ldots, M \).
For all sufficiently small $\Delta x$, $p_1^2 > 0$ and as in Lemma 1, $P$ symmetrizes $A$.

Let $\tilde{\sigma}_k = (\gamma_{k+1}^+ \beta_k)^{1/2}$, then

$$P^{-1}AP = \begin{bmatrix} \alpha_1 & \sigma_1 \\ \vdots & \ddots \\ \sigma_1 & \ddots & \sigma_{M-1} \\ \vdots & \ddots & \ddots & \sigma_{M-1} \\ \sigma_{M-1} & \ddots & \cdots & \alpha_M \end{bmatrix}$$

Observe that by the mean value theorem

$$\omega_k \omega_{k+1} = (a_{k+\frac{1}{2}})^2 \left[ 1 + O(\Delta x^2) \right] \text{ as } \Delta x \to 0$$

Also if $b(x) \in C^1[0,1]$, then

$$\tilde{\gamma}_{k+1}^+ \tilde{\beta}_k = \frac{(a_{k+\frac{1}{2}})^2 + a_{k+\frac{1}{2}} \left( b_k - b_{k+1} \right) \Delta x - \frac{b_k b_{k+1} \Delta x^2}{4}}{\omega_k \omega_{k+1}}$$

$$= \frac{(a_{k+\frac{1}{2}})^2 \left[ 1 + O(\Delta x^2) \right]}{(a_{k+\frac{1}{2}})^2 \left[ 1 + O(\Delta x^2) \right]} \text{ as } \Delta x \to 0$$

Hence,

$$\sigma_k = (\tilde{\gamma}_{k+1}^+ \tilde{\beta}_k)^{1/2} = 1 + O(\Delta x^2) \text{ as } \Delta x \to 0$$

Let $V = P^{-1} W^j$ and write the system (4.5) as
\[- \left[ 2 + \left( \frac{c_k - \Lambda_j}{\omega_k} \right) \Delta x^2 \right] v_k + \sigma_k v_{k+1} + \sigma_k v_{k-1} = 0 \]

(4.11)

\[v_0 = v_{M+1} = 0 \quad K = 1, \cdots, M\]

Let \( K_1 \) and \( K_2 \) be the constants in Lemma 3 and define

(4.12) \[\beta_j^2 = \frac{\Lambda_j}{K_2}\]

Let \( y(x) = \sin \beta_j x \). Then \( y_k = y(k\Delta x) \) satisfies the difference equations.

(4.13) \[- \left[ 2 - \mu_j \Delta x^2 \right] y_k + y_{k+1} + y_{k-1} = 0 \quad k = 1, 2, \ldots.\]

where

(4.14) \[\mu_j = \frac{4}{\Delta x^2} \sin^2 \frac{\beta_j \Delta x}{2}\]

The distance between successive zeros of \( y(x) \) is \[\frac{\pi}{\beta_j} = \frac{K_2 \pi^2}{\Lambda_j} \geq \frac{1}{j}\]

for \( j \) large enough by Lemma 3.

Let \( v(x) \) be the piecewise linear function corresponding to "graph" of vector \( V = P^{-1} W^j \). Define the auxiliary function \( z(x) \) by

\[z(x) = \frac{v(x)}{v(x)} \text{ whenever } v(x) \neq 0\]

We proceed to estimate the distance between successive nodes of \( v(x) \) by investigating the difference equation satisfied by \( z(x) \).
We may assume that \( \delta_{\text{Max}}(V) > 3\Delta x \). For if \( \delta_{\text{Max}}(V) \leq 3\Delta x \), then in particular, \( \delta_{\text{Max}}(V) \leq \frac{3}{M+1} < \frac{3}{j} \leq 3\pi \frac{K_2}{\Lambda_j} \) for all sufficiently large \( j \).

If \( \delta_{\text{Max}} > 3\Delta x \), then there exists a set \( N \) of consecutive mesh points, containing at least three members on which \( v(x) \) is strictly positive (or strictly negative). Let \( N' \) be \( N \) minus the 2 end points of \( N \). Since \( z_k = \frac{v_k}{v_k} \) for \( k \in N' \),

\[
(4.17) \quad \left[ \ell_h z \right]_k = - \left[ \frac{(2 - \mu_j \Delta x^2)\sigma_k}{2 + (c_k - \Lambda_j)\Delta x^2} \left( v_{k+1} + v_{k-1} \right) \right] \frac{z_k}{\omega_k} + v_{k+1} z_{k+1} + v_{k-1} z_{k-1} = 0 \quad k \in N'.
\]

We now show that for all sufficiently large \( j \), the difference operator \( \ell_h \) (or \( -\ell_h \) if \( v \) is strictly negative) occurring in (4.17) is of positive type, and hence satisfies the discrete maximum principle:

It is sufficient to show that if \( j \) is sufficiently large,

\[
(4.18) \quad \frac{[2 - \mu_j \Delta x^2] \sigma_k}{2 + (c_k - \Lambda_j)\Delta x^2} \geq 1 \quad \text{if} \quad k \in N'.
\]

From (4.14) we have \( \mu_j \leq \frac{\Lambda_j}{K_2} \leq \frac{\Lambda_j}{2a_1} \) if \( K_2 \) is chosen so that \( K_2 \geq 2a_1 \), where \( a_1 \) is an upper bound for \( a(x) \) on \([0,1]\). Hence,

\[
(4.19) \quad (2 - \mu_j \Delta x^2) \sigma_k = 2 - \mu_j \Delta x^2 + O(\Delta x^2)
\]
since \( \mu_j \Delta x^2 \leq 4 \) and \( \sigma_k = 1 + O(\Delta x^2) \).

Now,

\[
2 - \mu_j \Delta x^2 + O(\Delta x^2) \geq 2 - \frac{\Lambda_j \Delta x^2}{K_2}
\]

\[
\geq 2 - \frac{\Lambda_j \Delta x^2}{2\omega_k} + O(\Delta x^2)
\]

\[
= 2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k} + \frac{2(\Lambda_j - 2c_k)\Delta x^2}{2\omega_k} + O(\Delta x^2)
\]

i.e.

\[ (2 - \mu_j \Delta x^2) \sigma_k \geq 2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k} \quad \text{if } j \text{ is sufficiently large, since we assume } c(x) \text{ is bounded.} \]

Furthermore \( 2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k} \) is positive for \( K \in \mathbb{N}^+ \) since \( v_k, v_{k+1}, v_{k-1} \) have the same sign, on using (4.11). Thus (4.18) is satisfied.

Suppose now that \( z(x) \) has two zeroes in the interval spanned by \( N \).

At any mesh point lying between the two zeroes we must have \( z(x) = 0 \) by the maximum principle. Since \( z(x) = 0 \) if and only if \( y(x) = 0 \), this means that the distance between successive zeroes of \( y(x) \) is \( \leq \Delta x = \frac{1}{M+1} \).

However, as already noted, this distance is \( \geq \frac{1}{j} \) and \( j \leq M \).

Thus \( y(x) \) has at most one zero in the interval spanned by \( N \).

Hence the maximum distance between successive modes of \( v(x) \) must be less than or equal to \( \frac{\pi}{\beta_j} + 2\Delta x \).
Since $\Lambda_j = O(\frac{1}{\Delta x^2})$, we have

\[(4.21) \quad \delta_{\text{Max}}(V) \leq K(\Lambda_j)^{-\frac{1}{2}}\]

A similar estimate is valid for the eigenvector $W^j$ of $L$ since $W^j = PV$ and $P$ is a positive diagonal matrix. Q.E.D.

5. REMARKS

(a) It seems plausible that one also has an estimate

\[(4.22) \quad \delta_{\text{Min}} \geq K_0(\Lambda_j)^{-\frac{1}{2}} \quad \text{for } j \text{ large for the minimum distance between the nodes of } W^j.\]

(b) The estimate in Theorem 2 may be combined with Lemma 3 to show that if the eigenvectors $\{V^p\}$ of $L$ are normalized so that $\|V^p\|_2 = 1$, then

\[(4.23) \quad \|V^p\|_\infty \leq K_1 p^{\frac{1}{2}} \quad \text{for all sufficiently large } p. \quad (\text{See } [2]). \quad \text{Such an estimate was obtained by B"{u}ckner in the self-adjoint case using an elementary device ([[1]]).} \quad \text{Combined with Lemma 3, (4.23) shows that}

\[\sum_{\Lambda_p} \|V^p\|_\infty \quad \text{remains bounded as } M \to \infty.\]
REFERENCES


