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BOUNDARY CONTROLLABILITY OF NONLINEAR HYPERBOLIC SYSTEMS

by

Marco Cirinà

Technical Report #40

October, 1968

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1. INTRODUCTION

Russell has shown in [12] that if $T > 0$ is a real number not too small, it is possible to find a real valued function $u = u(t)$ on $[0,T]$, called boundary control, such that there is a function $w = w(x,t)$ defined on $\mathcal{R} = [0,1] \times [0,T]$ which satisfies

(1.1) $w_{tt} = c(x)\, w_{xx}$, $(x,t) \in \mathcal{R}$

(1.2) $w(x,0) = f(x)$, $w_t(x,0) = h(x)$, $x \in [0,1]$

(1.3) $w(0,t) = 0$, $w_x(1,t) = u(t)$, $t \in [0,T]$

(1.4) $w(x,T) = 0$, $x \in [0,1]$,

where c, f, h are appropriately given functions. Results on the boundary controllability of linear equations are also given in [1] for the simplest wave equation, and in [5] for general equations including the case of many space variables. See also [9], [3].

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* Supported in part by the National Science Foundation under Grant GP 6070

**Present address: Department of Mathematics, New York University, New York, N.Y.

In this paper we prove the existence of boundary controls for nonlinear first order hyperbolic systems in two independent variables, which include the system formulation of (1.1) as a particular case. The data and the coefficient of the equations will always be assumed to have continuous first derivatives and so the solutions will be C^1 functions satisfying the equations everywhere on their domain of definition. One important difference between linear and nonlinear hyperbolic problems is that conditions insuring the existence of a local solution are not sufficient, in general, to guarantee the existence of a solution of the latter problems on a set of preassigned size. On the other hand, in a initial value problem the data determine the solution completely on a certain set. Hence, in the nonlinear case, for the existence of boundary controls it is necessary that the solution of the relevant hyperbolic problem can be extended to sets of given size.

Some such extensions have been studied in [2], and the conditions found there, strengthened so as to make the time-like and the space-like variables interchangeable, will be seen to be sufficient for the existence of boundary controls in the nonlinear case. Also for the analogue of (1.1) to (1.4) it will be proved that if T is not too small, then in an appropriate Banach space there is an open set of initial data which can be brought to zero in time T.

2. DEFINITIONS, ZERO CONTROLLABILITY

Let $m > 0$, $n > 0$ be integers; R, R_m, $R_{m \times n}$ are respectively the real numbers, the m-dimensional euclidean space and the space of real matrices with m rows and n columns.

NORMS Throughout this paper, $|\cdot|$ denotes sup-norms; so $|h|$ = absolute value of h if $h \in R$, $|h| = \max\{|h_i| : i=1,\ldots,m\}$ if $h = (h_i) \in R_m$, $|h| = \max\{\sum_{j=1}^{n} |h_{ij}| : i=1,\ldots,m\}$ if $h = (h_{ij}) \in R_{m \times n}$, and $|h| = \sup\{|h(x)| : x \in X\}$ if h is a function defined on a set X and taking values in either R, or R_m, or $R_{m \times n}$.

From now on $\overline{m}$, $\underline{m}$ denote positive integers. Fix $m = \overline{m} + \underline{m}$; if $h = (h_i) \in R_m$, $\overline{h}$ and $\underline{h}$ are the points of $R_{\overline{m}}$, $R_{\underline{m}}$ whose components are defined by

$$\overline{h}_i = h_i \quad i=1,\ldots,\overline{m}\ ;\ \underline{h}_i = h_{i+m} \quad i=1,\ldots,\underline{m}\ ;$$

analogously if $h = (h_{ij}) \in R_{m \times m}$, $\overline{h}$ is the submatrix of h formed by the first $\overline{m}$ rows of h, and $\underline{h}$ that formed by the last $\underline{m}$ rows; if h is a function taking values in R_m or $R_{m \times m}$, $\overline{h}$ and $\underline{h}$ are defined similarly. If α is a positive real and $\mathcal{R} \subset R_2$, $\mathcal{R}_\alpha$ is the set defined by

$$\mathcal{R}_\alpha = \{(x,t,w) : (x,t) \in \mathcal{R}\ ,\ w \in R_m, |w| \leq \alpha\}\ ;$$

if $a, T \in (0,\infty]$, $\mathcal{R} = \mathcal{R}(a,T)$ is the set

$$\mathcal{R} = \{(x,t) : 0 \leq x \leq a\ ,\ x \neq \infty,\ 0 \leq t \leq T,\ t \neq \infty\};$$

In (iii) and throughout this paper S^{-1} is the map $(x,t,w) \to (S(x,t,w))^{-1}$, where the last object is the inverse of $S(x,t,w)$ in $R_{m\times m}$; A^{-1} and D^{-1} are defined analogously. Note that (i) (i)' amount to the definition of "the system (2.1) is hyperbolic on $\mathfrak{R}_\alpha$", and (ii) (ii)' are usual conditions in dealing with hyperbolic mixed boundary problems; also if $\mathfrak{R}$ is compact, (iii) is redundant and it suffices that (i) (ii) (ii)' hold for $\delta = 0$.

THE CLASS $C^1(X,Y)$ We write $z \in C^1(X,Y)$ as an abbreviation of "z is a continuously differentiable Y-valued function on X"; in absence of ambiguity the range space Y will be omitted. If X is an interval, $C^1(X,R_m)$ is given the following more special meaning. Let $I \in R$ be a compact interval; $C^1(I,R_m)$ is the set of R_m-valued continuous functions ϕ on I possessing a continuous derivative on I, endowed with its usual algebraic structure and normed by

$$\|\phi\| = |\phi| + |\phi'|,$$

where ϕ' is the derivative of ϕ. $C^1_o = C^1_o([0,1],R_m)$ is the subspace of $C^1 = C^1([0,1],R_m)$ defined by

$$C^1_o = \{\phi : \phi \in C^1,\ \phi(0) = \phi'(0) = 0\}.$$

So C^1, C^1_o are both Banach spaces.

ZERO CONTROLLABILITY Put $\mathfrak{R} = \mathfrak{R}(1,\infty)$; suppose $A = A(x,t,w)$, $f = f(x,t,w)$ are functions defined on $\mathfrak{R}_\alpha$, A is $R_{m\times m}$ valued,

f is R_m valued. We say that the system

(2.1) $z_t + A(x,t,z)z_x = f(x,t,z)$

is <u>zero controllable with one boundary control</u> if $m = \overline{m} + \underline{m}$, some $\overline{m}$ $\underline{m}$, and there is an open set $\Omega \in C^1_o([0,1],R_m)$ such that for each $\phi \in \Omega$ there exist a real number $T > 0$ and a function $\underline{u} = \underline{u}(t)$ from $[0,T]$ to $R_{\underline{m}}$ so that the solution $z = z(x,t)$ of

(2.2) $z_t + A(x,t,z)z_x = f(x,t,z)$, $(x,t) \in \mathcal{R}\ (1,T)$

(2.3) $z(x,0) = \phi(x)$, $0 \leq x \leq 1$

(2.4) $\overline{z}(0,t) = 0$, $\underline{z}(1,t) = \underline{u}(t)$, $0 \leq t \leq T$

exists and satisfies

(2.5) $z(x,T) = 0,\ 0 \leq x \leq 1.$

Analogously, we say that (2.1) is <u>zero controllable with two boundary controls</u> if $m = \overline{m} + \underline{m}$, some $\overline{m}$ $\underline{m}$, and there is an open set $\Omega \in C^1([0,1],R_m)$ such that for each $\phi \in \Omega$ there exists a real number $T > 0$ and functions $\overline{u} = \overline{u}(t)$, $\underline{u} = \underline{u}(t)$ from $[0,T]$ to $R_{\overline{m}}$ and $R_{\underline{m}}$ respectively, so that the solution $z = z(x,t)$ of

(2.6) $z_t + A(x,t,z)z_x = f(x,t,z)$, $(x,t) \in \mathcal{R}(1,T)$

(2.7) $z(x,0) = \phi(x)$, $0 \leq x \leq 1$

(2.8) $\overline{z}(0,t) = \overline{u}(t)$, $\underline{z}(1,t) = \underline{u}(t)$, $0 \leq t \leq T$,

exists and satisfies

$$z(x,T) = 0 \ , \ 0 \leq x \leq 1.$$

3. THE MAIN RESULT

If A is of class $\overline{\Sigma}$ then (3.1) is zero controllable with one boundary control. This assertion is a particular case of our main result, theorem 3.III, which will be seen to follow mainly from the fact that if $A \in \Sigma$ then the solution of (3.1) to (3.3) below exists on a preassigned rectangle whenever the data $\phi\ \overline{u}\ \underline{u}$ are conveniently restricted. For $m = \overline{m} + \underline{m}$ consider the following mixed boundary problem

(3.1) $z_t + A(x,t,z)z_x = 0 \;,\; (x,t) \in \mathcal{R}(a,T)$

(3.2) $z(x,0) = \phi(x) \;,\; x \in [0,a]$

(3.3) $\overline{z}(0,t) = \overline{u}(t) \;,\; \underline{z}(a,t) = \underline{u}(t) \;,\; t \in [0,T]$

where

(3.4) $\phi \in C^1([0,a],R_m) \;,\; \overline{u} \in C^1([0,T],R_{\overline{m}}) \;,\; \underline{u} \in C^1([0,T],R_{\underline{m}})$

and $\phi\ \overline{u}\ \underline{u}$ satisfy the compatibility conditions

(i) $\overline{u}(0) = \overline{\phi}(0) \;,\; \overline{u}'(0) + \overline{A}(0,0,\phi(0))\phi'(0) = 0$

(ii) $\underline{u}(0) = \underline{\phi}(a) \;,\; \underline{u}'(0) + \underline{A}(a,0,\phi(a))\phi'(a) = 0.$

3.I REMARK Suppose z satisfies (3.1), $0 \leq t < T$, and for $i=1,\ldots,\overline{m}$ let $\xi_i = \xi_i(s)$ be defined by

$$\frac{d}{ds}\xi_i(s) = d_i(\xi_i(s),s,z(\xi_i(s),s)) \;,\; s > t,\; \xi_i(t) = 0,$$

where d_i is the ith diagonal element of $D = SAS^{-1}$. If $d_i > 0$ then at $s = t$ the curve $(\xi_i(s),s)$ $s \geq t$, called the ith characteristic of (3.1) through $(0,t)$, enters the rectangle $\mathcal{R} = \mathcal{R}(a,T)$. So if $A \in \overline{\Sigma}\ (\mathcal{R}, m, \alpha)$, the first of (3.3) amount to fixing on the boundary $x = 0$ of $\mathcal{R}$, exactly those components of z which correspond to characteristics entering $\mathcal{R}$ there.

The following continuation result is known, see [2] theorems 5.III and 5.I.

3.II LEMMA Fix $0 < c_o < \alpha$, $0 < T < \infty$ all real, $m = \overline{m} + \underline{m}$, $0 < \epsilon < b \leq \infty$, $\mathcal{R} = \mathcal{R}\ (b, \infty)$ and $A = S^{-1}DS \in \overline{\Sigma}\ (\mathcal{R}, m, \alpha)$.
Conclusion: there are real numbers $c > 0$, $N > 0$ such that if $a \in R \cap [\epsilon, b]$, $\phi\ \overline{u}\ \underline{u}$ satisfy (3.4), $|\phi| \leq c_o$ and $\max(|\phi'|, |\overline{u}'|, |\underline{u}'|) \leq c$ then on $\mathcal{R}\ (a, T)$ there is a (unique) function z of class C^1 which satisfy (3.1) to (3.3), and $|z_x| \leq N(|\phi'| + |\overline{u}'| + |\underline{u}'|)$; moreover for $0 < T_1 \leq \min(T, \frac{a}{|\underline{D}|})$ the restriction of z to the triangle $\tau(a, T_1)$ is independent of the choice of $\underline{u}$.

Lemma 3.II is the main tool for proving the following

3.III THEOREM Put $\mathcal{R} = \mathcal{R}\ (1, \infty)$; fix $m = \overline{m} + \underline{m}$, $A = S^{-1}DS \in \overline{\Sigma}(R, m, \alpha)$, $0 < c_o < \alpha$ and $\overline{u} \in C^1([0, \infty], R_{\overline{m}})$ with bounded support.
Conclusion: there is $c > 0$ real such that if $\phi \in C^1([0, 1], R_m)$, $\phi\ \overline{u}$ satisfy (3.4) (i), $|\phi| \leq c_o$ and $\max(|\phi'|, |\overline{u}'|) \leq c$, then there exist $0 < T < \infty$, $\underline{u} \in C^1([0, T], R_{\underline{m}})$ so that the solution $z = z(x, t)$ of

(3.1) to (3.3) with $a = 1$ exists in $C^1(\mathcal{R}(1,T), R_m)$ is unique there, and moreover satisfies

(1) $z(x,T) = 0$, all $x \in [0,1]$; $|z| \leq \min(\alpha, 2C_o)$

PROOF Fix c_1, c_2, t_o real so that

(2) $c_o < c_1 < c_2 < \min(\alpha, 2c_o)$; $\overline{u}(t) = 0$ all $t \geq t_o \geq 0$; and define

(3) $T_o = |\underline{D}|^{-1}$, $T_1 = |\underline{(D^{-1})}|$;

note that T_o, T_1 are real, positive and $T_o \leq T_1$.

For each real $\delta > 0$, fix a real number $\Delta = \Delta(\delta)$ such that if h satisfies

(4) $h \in C^1([0,T_1], R_{\overline{m}})$, $|h(T_1)| \leq c_1$, $|h'(T_1)| \leq \delta$

then h has a C^1 extension H to $[0,\infty)$ satisfying $|H| \leq c_1 + \delta$, $|H'| \leq \delta$, $H(t) = 0$ all $t \geq T_1 + \Delta$.

(α1) Consider the mixed boundary problem

(5) $z_x + A^{-1}(x,t,z)z_t = 0$, $(x,t) \in \mathcal{R}(1,T)$

(6) $z(0,t) = \psi(t)$, $t \in [0,T]$

(7) $\overline{z}(x,0) = \overline{\phi}(x)$, $\underline{z}(x,T) = 0$, $x \in [0,1]$.

As it is easily checked $A^{-1} \in \overline{\Sigma}(\mathcal{R}, m, \alpha)$; hence by 3.II there is δ_2

(8) $0 < \delta_2 \leq \min(\frac{c_1 - c_o}{T_1}, c_2 - c_1)$

such that if

(9) $T_o \leq T < \infty$, $\psi \in C^1([0,T], R_m)$; $\psi(T) = \psi'(T) = 0$,

$\bar{\psi}(0) = \bar{\phi}(0)$, $\bar{\phi}'(0) + \overline{A^{-1}}(0,0,\psi(0))\psi'(0) = 0$;

$|\psi| \leq c_2$ and $\max(|\psi'|, |\bar{\phi}'|) \leq \delta_2$

there is a (unique) $z \in C^1((1,T))$ satisfying (5) to (7) and

(10) $|z| \leq \min(\alpha, 2c_o)$.

(α2) For $\underline{v} = \underline{v}(t)$ satisfying

(11) $\underline{v} \in C^1([0,T_1], R_{\underline{m}})$, $v(0) = \underline{\phi}(1)$, $\underline{v}'(0) + \underline{A}(1,0,\phi(1))\phi'(1) = 0$

consider the mixed boundary problem

(12) $w_t + A(x,t,w)w_x = 0$, $(x,t) \in \mathcal{R}(1,T_1)$

(13) $w(x,0) = \phi(x)$, $x \in [0,1]$

(14) $\bar{w}(0,t) = \bar{u}(t)$, $\underline{w}(1,t) = \underline{v}(t)$, $t \in [0,T_1]$.

Since $A \in \bar{\Sigma}(\mathcal{R}, m, \alpha)$, Lemma 3.II implies that there is $\delta_1 > 0$ such that whenever

$\phi, \bar{u}, \underline{v}$ satisfy (3.4) for $a = 1$ $T = T_1$, (11) , $|\phi| \leq c_o$

and $\max(|\phi'|, |\bar{u}'|, |\underline{v}'|) \leq \delta_1$

there is a function $w \in C^1(\mathcal{R}(1,T_1), R_m)$ satisfying (12), (13), (14) and

(15) $|w_t| \leq \delta_2$, $|w| \leq c_1$.

Hence there is c, $0 < c \leq \delta_2$, so that if $\phi, \bar{u}$ are as in the hypotheses of the theorem, there is v satisfying (11) and (a unique) $w \in C^1(\mathcal{R}(1,T_1),R_m)$ satisfying (12) to (15). Indeed c can be taken to be any number satisfying

$$0 < c \leq \min(\delta_2,\delta_1) \ , \ |\underline{A}| \ c \leq \delta_1,$$

and $\underline{v}$ any function satisfying (11) and $|\underline{v}'| \leq c$, for instance

$$\underline{v}(t) = \underline{\phi}(1) + t \ \underline{A}(1,0,\phi(1))\phi'(1).$$

It will now be proved that for this c the conclusion of the theorem holds. To this end, let $\phi, \bar{u}$ be as in the hypotheses; fix $\underline{v}$ as said in (α2) and let w be the function satisfying (12) to (15).

Define

$$T_2 = T_1 + \Delta(\delta_2) \ ; \ h(t) = \underline{w}(0,t) \ , \ t \in [0,T_1] \ ;$$

in view of (8) and the definition of $\Delta(\delta_2)$, h has a C^1 extension H to $[0,\infty]$ with $|H| \leq c_2$, $|H'| \leq \delta_2$ and $H(t) = 0$ all $t \geq T_2$.

Let

$$T_2' = \max(T_2,t_o) \ ; \ T = T_2' + |\overline{(D^{-1})}|$$

and define the function $\psi = \psi(t)$ by

$$\bar{\psi}(t) = \bar{u}(t) \ , \ \underline{\psi}(t) = H(t) \ , \ t \in [0,T] \ ;$$

then $\psi(t) = 0$ for $T_2' \leq t \leq T$ and ψ satisfies (9).

So let $z^* = z^*(x,t)$ be the only function in $C^1(\mathcal{R}(1,T),R_m)$ satisfying (5) (6) (7) (10).

It will now be shown that z^* satisfies also

(16) $\quad z^*(x,T) = 0$, all $x \in [0,1]$

(17) $\quad z^*(x,0) = \phi(x)$, all $x \in [0,1]$.

To this end let $\tau_o \subset \mathcal{R}(1,T)$ be the triangle

$$\tau_o = \{(x,t) : 0 \leq x \leq 1 \ , \ T_2' + (T_2'-T)x \leq t \leq T\}$$

and consider the mixed boundary problems

(18) $\quad z_x + A^{-1}(x,t,z)z_t = 0$

(19) $\quad z(0,t) = 0$, $t \in [T_2',T]$

(20) $\quad \underline{z}(x,T) = 0$, $x \in [0,1]$;

the zero function on τ_o and the restriction of z^* to τ_o both satisfy (18) on τ_o, (19) and (20); also since $T - T_2' = |\overline{(D^{-1})}|$, the last assertion in 3.II implies that on τ_o there is at most one function in $C^1(\tau_o,R_m)$ which satisfies (21) to (23); hence (16) holds because

$$z^*(x,t) = 0 \ , \text{ all } (x,t) \in \tau_o .$$

Analogously, let

$$\tau = \{(x,t) : 0 \leq x \leq 1 \ , \ 0 \leq t \leq T_1 - T_1 x\}$$

and consider the mixed boundary problem

(21) $\quad z_t + A^{-1}(x,t,z)z_x = 0$

(22) $\quad z(0,t) = \psi(t)$, $t \in [0,T_1]$

(23) $\quad \overline{z}(x,0) = \overline{\phi}(x)$, $x \in [0,1]$;

In view of the definitions of ψ, w, z*, it is easily seen that the restrictions $w|_\tau$ and $z^*|_\tau$ both belong to $C^1(\tau, R_m)$ and satisfy (21) on τ, (22), (23); since $T_1 = |\underline{(D^{-1})}|$, on τ uniqueness prevails, and hence

$$w|_\tau = z^*|_\tau ;$$

this proves (17).

Define

$$\underline{u}(t) = \underline{z}^*(1,t) \ , \ t \in [0,T] ;$$

then $\underline{u} \in C^1([0,T], R_{\underline{m}})$. Since $z = z^*$ satisfies (5) to (7), (10), (16), (17), from the definition of ψ, $\underline{u}$ follows that z* is the solution of (3.1) to (3.3) with $a = 1$, and satisfies (1). This completes the proof.

4. CONTROLLABILITY OF $z_t + A(x,t,z)z_x = 0$

Let us first note that the existence proof of the boundary control $\underline{u}$, as given in 3.III, is constructive and, as it will be indicated later in this section, it is well suited as a basis for the numerical computation of such control. Some consequences of the main result will now be made explicit. The special case of 3.III for $\bar{u} = 0$ can be restated as

4.I THEOREM Suppose $\mathcal{R} = \mathcal{R}(1,\infty)$, $m = \bar{m} + \underline{m}$ and $A \in \bar{\Sigma}(\mathcal{R}, m, \alpha)$. Then $z_t + A(x,t,z)z_x = 0$ is zero controllable with one boundary control.

Fix m, c_o as in 3.III and let $c > 0$ be as given there for $\bar{u} = 0$; put $C^1 = C^1([0,1], R_m)$ and define $\Omega_o = \Omega_o(c_o, c)$ by

$$\Omega_o = \{\phi : \phi \in C^1,\ \bar{\phi}(0) = 0\ ,\ |\phi| \leq c_o,\ |\phi'| \leq c\}.$$

Then Ω_o contains non trivial open sets of $C^1_o([0,1], R_m)$, and from the proof of 3.III it is clear that the real number $T > 0$ produced there does not depend on the choice of ϕ in Ω_o, i.e. any ϕ in Ω can be brought to zero in time T; furthermore T cannot be too small. This is formalized in the following

4.II COROLLARY Suppose the hypotheses of 4.I hold, and fix $0 < c_o < \alpha$. Then

(i) there are real numbers $c < 0$ $T < 0$ such that if $\phi \in \Omega_o(c_o, c)$ there exist $\underline{u} \in C^1([0,T], R_{\underline{m}})$ and $z \in C^1(R(1,T), R_m)$ satisfying

(1) $z_t + A(x,t,z)z_x = 0$, $(x,t) \in \mathcal{R}(1,T)$

(2) $z(x,0) = \phi(x)$, $x \in [0,1]$

(3) $\overline{z}(0,t) = 0$, $\underline{z}(1,t) = \underline{u}(t)$, $t \in [0,T]$

(4) $z(x,T) = 0$, $x \in [0,1]$;

(ii) if c, T is any such pair, then $T \geq |\underline{D}|^{-1}$;

(iii) if c_o is sufficiently small, there are c, T having the properties in (i) and in addition

$$T \leq |\underline{(D^{-1})}| + 1 + |\overline{(D^{-1})}|.$$

PROOF (i) has already been seen, (iii) follows immediately from the proof of 3.III for $\overline{u} = 0$, and to establish (ii) it suffices to notice that in $\Omega_o(c_o, c)$ there are initial data (for instance $\overline{\phi} = 0$, $\underline{\phi} = \frac{c_o}{2}$) for which on the triangle $\tau(1, |\underline{D}|^{-1})$ the solution of (1) (2) and the first part of (3) is a non zero constant.

4.III REMARK For a given $\phi \in \Omega(c_o, c)$ the control function $\underline{u}$ is by no means unique. This is due to the fact that in the construction of $\underline{u}$, see proof of 3.III, one can choose $\underline{v}$ among infinitely many functions and extend $h(t) = \underline{w}(0,t)$ in infinitely many ways. For instance it is easily seen that h can be usefully extended by using any function in some closed convex set contained in $C^1([T_1,T_2],R_{\underline{m}})$ and containing more than one element, hence infinitely many; also in the proof of 3.III it is shown that to each such extension H of h there correspond $\underline{u} \in C^1([0,T],R_{\underline{m}})$ such that (1) to (4) in 4.II hold; on the other hand from the uniqueness of solution of the mixed boundary problem (1) to (3) in 4.II follows that the map $H \to \underline{u}$ is one to one; whence there are infinitely many $\underline{u}$ which bring the given ϕ to zero in finite time.

It will now be shown that the hyperbolic system studied so far is also zero controllable with two boundary controls. This is a consequence of the continuation result 3.II and the proof of 3.III. For $c_o > 0$ $c > 0$ real and $C^1 = C^1([0,1],R_m)$ define $\Omega = \Omega(c_o,c)$, a subset of C^1 with non trivial interior, by

$$\Omega = \{\phi : \phi \in C^1, \ |\phi| \leq c_o \ , \ |\phi'| \leq c\}.$$

4.IV THEOREM Put $\mathfrak{R} = \mathfrak{R}(1,\infty)$, fix $m = \overline{m} + \underline{m}$, $0 < c_o < \alpha < \infty$ and suppose $A = S^{-1}DS \in \overline{\Sigma}(\mathfrak{R},m,\alpha)$.

Conclusion: there is $c > 0$ real such that if $\phi \in \Omega(c_o,c)$ there are $0 < T < \infty$, $\overline{u} \in C^1([0,T],R_{\overline{m}})$, $\underline{u} \in C^1([0,T],R_{\underline{m}})$ so that the solution $z = z(x,t)$ of (3.1) to (3.3) with $a = 1$ exists in $C^1(\mathfrak{R}(1,T),R_m)$ and satisfies

(1) $z(x,T) = 0$, all $x \in [0,1]$; $|z| \leq \min(\alpha, 2c_o)$.

Thus $z_t + A(x,t,z)z_x = 0$ is zero controllable with two boundary controls.

PROOF Fix $c_o < c_1 < c_2 < \min(\alpha, 2c_o)$ and define

(2) $T_o = \frac{|D|^{-1}}{2}$, $T_1 = \frac{|D^{-1}|}{2}$

where, it is recalled, D^{-1} is the map $(x,t,w) \to (D(x,t,v))^{-1}$; so T_o, T_1 are real and $T_o \leq T_1$.

For each real $\delta > 0$, fix a real number $\Delta = \Delta(\delta)$ such that if h satisfies

(3) $h \in C^1([0,T_1], R_m)$, $|h| \leq c_1$, $|h'| \leq \delta$

then h has a C^1 extension H to $[0,\infty]$ satisfying

$|H| \leq c_1 + \delta$, $|H| \leq \delta$, $H(t) = 0$ all $t \geq T_1 + \Delta$.

(α1) Consider the pair of mixed boundary problems

(4) $z_x + A^{-1}(x,t,z)z_t = 0$, $(x,t) \in \mathcal{R}^- = [0,\frac{1}{2}] \times [0,T]$

(5) $z(\frac{1}{2},t) = \psi(t)$, $t \in [0,T]$

(6) $\underline{z}(x,0) = \underline{\phi}(x)$, $\overline{z}(x,T) = 0$, $x \in [0,\frac{1}{2}]$

(4') $z_x + A^{-1}(x,t,z)z_t = 0$, $(x,t) \in \mathcal{R}^+ = [\frac{1}{2},1] \times [0,T]$

(5') $z(\frac{1}{2},t) = \psi(t)$, $t \in [0,T]$

(6') $\overline{z}(x,0) = \overline{\phi}(x)$, $\underline{z}(x,T) = 0$, $x \in [\frac{1}{2},1]$.

Since $A^{-1} \in \overline{\Sigma}(\mathcal{R}, m, \alpha)$, lemma 3.II implies that there is δ_2

(7) $$0 < \delta_2 \leq \min\left(\frac{c_1 - c_o}{T_1},\ c_2 - c_1\right)$$

such that if

(8) $T \in [T_o, \infty)$, $\psi \in C^1([0,T], R_m)$, $\phi \in C^1([0,1], R_m)$,

ψ and ϕ satisfy the compatibility conditions $\psi(T) = \psi'(T) = 0$,

$\psi(0) = \phi(\frac{1}{2})$, $\phi'(\frac{1}{2}) + A^{-1}(\frac{1}{2}, 0, \phi(\frac{1}{2}))\psi'(0) = 0$, and

$|\psi| \leq c_2$, $\max(|\psi'|, |\phi'|) \leq \delta_2$

then there is a unique pair of functions $z_- \in C^1(\mathcal{R}^-, R_m)$ satisfying (4) to (6), $z_+ \in C^1(\mathcal{R}^+, R_m)$ satisfying (4') to (6') and moreover

(9) $$|z_-|,\ |z_+| \leq \min(\alpha, 2c_o).$$

(α2) For $\overline{v}$, $\underline{v}$ satisfying

(10) $$\overline{v} \in C^1([0,T_1], R_{\overline{m}}),\ \overline{v}(0) = \overline{\phi}(0),\ \overline{v}'(0) + \overline{A}(0,0,\phi(0))\phi'(0) = 0$$

$$\underline{v} \in C^1([0,T_1], R_{\underline{m}}),\ \underline{v}(0) = \underline{\phi}(1),\ \underline{v}'(0) + \underline{A}(1,0,\phi(1))\phi'(1) = 0$$

consider the mixed boundary problems

(11) $$w_t + A(x,t,z)w_x = 0,\ (x,t) \in \mathcal{R}(1, T_1)$$

(12) $$w(x,0) = \phi(x),\ x \in [0,1]$$

(13) $$\overline{w}(0,t) = \overline{v}(t),\ \underline{w}(1,t) = \underline{v}(t),\ t \in [0, T_1].$$

Since $A \in \overline{\Sigma}(\mathcal{R},m,\alpha)$, lemma 3.II, in view of the reasoning made in $(\alpha 2)$ of 3.III, implies that there is c, $0 < c < \delta_2$, such that if $\phi \in \Omega(c_o,c)$ there exist $\overline{v}$, $\underline{v}$ satisfying (10) and $w \in C^1(\mathcal{R}(1,T_1),R_m)$ satisfying (11) (12) (13) and

(14) $\quad |w_t| \leq \delta_2$, $|w| \leq c_1$.

To see that for this c the conclusion of the theorem holds, let $\phi \in \Omega(c_o,c)$, fix $\overline{v},\underline{v}$ so that what has been said in $(\alpha 2)$ holds, and let w be the function satisfying (11) to (14). Define

$$T_2 = T_1 + \Delta(\delta_2) \ , \ T = T_2 + \frac{|D^{-1}|}{2}$$

$$h(t) = w(\tfrac{1}{2},t) \ , \ t \in [0,T_1].$$

Since h satisfies (3), from the definition of $\Delta(\delta_2)$ and (7) follows that we can fix a function $\psi \in C^1([0,T],R_m)$ which extends h and satisfies $|\psi| \leq c_2$, $|\psi'| \leq \delta_2$ $\psi(t) = 0$ if $t \in [T_2,T]$. So T, ψ, ϕ satisfy (8); let z_-, z_+ be the solutions of (4) to (6) and (4') to (6') respectively. Define

$$\overline{u}(t) = \overline{z}_-(0,t) \ , \ \underline{u} = \underline{z}_+(0,t) \ , \ t \in [0,T];$$

then $\overline{u} \in C^1([0,T],R_{\overline{m}})$, $\underline{u} \in C^1([0,T],R_{\underline{m}})$. By using the same uniqueness arguments already used in the proof of 3.III, it follows that z_- is the solution of

$$z_t + A(x,t,z)z_x = 0 \ , \ (x,t) \in \mathcal{R}^-$$

$$z(x,0) = \phi(x) \ , \ x \in [0,\tfrac{1}{2}]$$

$$\overline{z}(0,t) = \overline{u}(t) \ , \ \underline{z}(\tfrac{1}{2},t) = \underline{\psi}(t) \ , \ t \in [0,T]$$

and satisfies

$$z_-(x,T) = 0 \ , \ x \in [0,\tfrac{1}{2}] \ ; \ |z_-| \leq \min(\alpha, 2c_o) \ ;$$

analogously, z_+ is the solution of

$$z_t + A(x,t,z)z_x = 0 \ , \ (x,t) \in \mathcal{R}^+$$

$$z(x,0) = \phi(x) \ , \ x \in [\tfrac{1}{2},1]$$

$$\overline{z}(\tfrac{1}{2},t) = \overline{\psi}(t) \ , \ z(1,t) = \underline{u}(t) \ , \ t \in [0,T]$$

and satisfies

$$z_+(x,T) = 0 \ , \ x \in [\tfrac{1}{2},1] \ ; \ |z_+| \leq \min(\alpha, 2c_o).$$

Define z to be z_- on $\mathcal{R}^-$, z_+ on $\mathcal{R}^+$; then $z \in C^1(\mathcal{R}(1,T),R_m)$, and a moment of reflection shows that z is the solution of (3.1) to (3.3) and satisfies (1). The theorem is thus established.

The two boundary controls $\overline{u}$, $\underline{u}$ are not unique; this depends, as before, on the fact that there are many useful choices of $\overline{v}$, $\underline{v}$ and many useful extensions of $w(\tfrac{1}{2},.)$. Incidentally this lack of uniqueness is most interesting since it leaves open the possibility of choosing the boundary controls $\overline{u}$ $\underline{u}$ so as to minimize T or, for fixed T, to minimize some functional of $\overline{u}$, $\underline{u}$ and z.

Next corollary is the analogue of 4.II and follows immediately from 4.IV. It asserts in particular that if the initial data ϕ have sufficiently small derivative and the real number $T > 0$ is not too small then ϕ can be brought to zero in time T.

4.V COROLLARY Let c_o, A be as in the hypotheses of 4.IV. Then

(i) there are real numbers $c > 0$, $T > 0$ such that for each $\phi \in \Omega(c_o, c)$ there exist $\overline{u} \in C^1([0,T], R_{\overline{m}})$, $\underline{u} \in C^1([0,T], R_{\underline{m}})$ and $z \in C^1(\mathcal{R}(1,T), R_m)$ satisfying

$$z_t + A(x,t,z)z_x = 0 \ , \quad (x,t) \in \mathcal{R}(1,T)$$

$$z(x,0) = \phi(x) \ , \quad x \in [0,1]$$

$$\overline{z}(0,t) = \overline{u}(t) \ , \quad \underline{z}(1,t) = \underline{u}(t) \ , \quad t \in [0,T]$$

$$z(x,T) = 0 \ , \quad x \in [0,1] \ ;$$

(ii) if c, T is any such pair, then $T \geq \dfrac{|D|^{-1}}{2}$;

(iii) if c_o is sufficiently small, there are c, T having the properties (i) and in addition

$$T \leq |D^{-1}| + 1 .$$

As for the numerical determination of the boundary controls it is useful to observe that the proofs of the existence theorems 3.III and 4.IV give a general method of computation. Indeed, in the case of one boundary control $\underline{u} = \underline{u}(t)$, the computation of $\underline{u}$ is reduced by 3.III to the numerical solution of two mixed boundary problems, namely (12) to (14) and (5) to (7) in 3.III. Analogously, in the case of two boundary controls, computation of $\overline{u}, \underline{u}$ is reduced by 4.IV to the numerical solution of three mixed boundary problems. Therefore any numerical scheme for solving hyperbolic mixed boundary problems, such as for instance those in [7], [8], [13], gives a scheme for computing boundary controls.

5. CONTROLLABILITY OF $z_t + A(x,t,z)z_x = f(t,z)$

It will be seen that sufficient conditions for the hyperbolic system

$$z_t + A(x,t,z)z_x = f(x,t,z)$$

to be zero controllable are the usual conditions on A, f for solving the mixed boundary problem, augmented by

$$(A^{-1}f)_t = 0, \quad f_x = 0, \quad \frac{|f(x,t,z)|}{|z|} \to 0 \quad \text{as} \quad z \to 0.$$

These additional requirements are used to guarantee that for some class of data the two relevant mixed problems analogous to (5) to (7) and (12) to (14) in theorem 3.III, have solution on preassigned rectangles. Since the system studied in section 4 satisfies the above additional conditions, the results in this section generalize those already obtained; however in a sense, they are also more special because the set of initial data ϕ brought to zero in finite time will be smaller, for not only $|\phi'|$ but also $|\phi|$ will be required to be small.

For α, T positive real define

$$B_\alpha = \{w : w \in R_m, \ |w| \leq \alpha\}.$$

Suppose

(5.1) $f = f(t,w)$ is a C^1 function from $[0,T] \times B_\alpha$ to R_m and for each $t \in [0,T]$, $\dfrac{|f(t,w)|}{|w|} \to 0$ as $w \to 0$;

consider the mixed boundary problem

(5.2) $z_t + A(x,t,z)z_x = f(t,z)$, $(x,t) \in \mathcal{R}(a,T)$

(5.3) $z(x,0) = \phi(x)$, $x \in [0,a]$

(5.4) $\overline{z}(0,t) = \overline{u}(t)$, $\underline{z}(0,t) = \underline{u}(t)$, $t \in [0,T]$

where

(5.5) $\phi \in C^1([0,a],R_m)$, $\overline{u} \in C^1([0,T],R_{\overline{m}})$, $\underline{u} \in C^1([0,T],R_{\underline{m}})$, and

(i) $\overline{u}(0) = \overline{\phi}(0)$, $\overline{u}'(0) + \overline{A}(0,0,\phi(0))\phi'(0) = f(0,\phi(0))$,

(ii) $\underline{u}(0) = \underline{\phi}(a)$, $\underline{u}'(0) + \underline{A}(a,0,\phi(a))\phi'(a) = f(0,\phi(a))$.

The following analogue of lemma 3.II is known, see [2] theorems 5.II and 5.I.

5.1 LEMMA Fix $m = \overline{m} + \underline{m}$, $0 < T < \infty$, $0 < b \leq \infty$, $A = S^{-1}DS \in \dot{\overline{\Sigma}}(\mathcal{R},m,\alpha)$ where $\mathcal{R} = \mathcal{R}(b,T)$, f satisfying (5.1), $0 < \epsilon < b$ and $N > 0$ real.

Conclusion: there are real numbers $c_o > 0$, $c > 0$ such that if $a \in R \cap [\epsilon,b]$, ϕ $\overline{u}$ $\underline{u}$ satisfy (5.5), $|\phi| \leq c_o$ and $\max(|\phi'|,|\overline{u}'|,|\underline{u}'|) \leq c$, then there is a unique $z \in C^1(\mathcal{R}(a,T),R_m)$ which satisfies (5.2) to (5.4), and moreover

$$|z| < 2c_o \ , \ |z_x| \leq N;$$

also if $0 < T_1 \leq \min(T, \frac{a}{|D|})$, the restriction of z to the triangle $\tau(a,T_1)$ does not depend on the choice of $\underline{u}$.

DEFINITION Suppose $0 < T \leq \infty$, $0 < b \leq \infty$ and $\mathcal{R} = \mathcal{R}(b,T)$; write $(A,f) \in \tilde{\Sigma}(\mathcal{R},m,\alpha)$ if and only if $A = A(x,t,w) \in \overline{\Sigma}(\mathcal{R},m,\alpha)$, $f = f(t,w)$ satisfies (5.1) with $[0,T]$ replaced by $[0,T] \cap R$, and $(A^{-1}f)_t = 0$.

5.II REMARK If $\mathcal{R}_\alpha$ is a convex set and the partial derivative of $(A^{-1}f)$ with respect to t vanishes everywhere on $\mathcal{R}_\alpha$, which is trivially true if A and f are independent of t, then

$$A^{-1}(x,t,w)f(t,w) = A^{-1}(x,\tilde{t},w)f(\tilde{t},w) \ , \ \text{all } (x,t,w),(x,\tilde{t},w) \in \mathcal{R}_\alpha .$$

Hence if $(A,f) \in \tilde{\Sigma}(\mathcal{R},m,\alpha)$, $A^{-1}f$ can be identified with the map $\tilde{f} = \tilde{f}(x,w)$ defined by

$$\tilde{f}(x,w) = A^{-1}(x,0,w)f(0,w) \ , \ (x,w) \in ([0,b] \cap R) \times B_\alpha ,$$

and $\tilde{f}$ satisfies the analogue of (5.1), i.e. $\tilde{f}$ is of class C^1 and for each $x \in [0,b] \cap R$, $\frac{|\tilde{f}(x,w)|}{|w|} \to 0$ as $w \to 0$. So if $(A,f) \in \tilde{\Sigma}(\mathcal{R},m,\alpha)$, A,f satisfy the hypotheses of 5.I, and A^{-1},f satisfy the hypotheses of 5.I with x playing the role of t; hence Lemma 5.I, rewritten with the obvious change in notation, holds for the mixed boundary problem

$$z_x + A^{-1}(x,t,z)z_t = f(x,z) \ , \ (x,t) \in \mathcal{R}(a,T)$$

$$z(0,t) = \psi(t), \ t \in [0,T]$$

$$\overline{z}(x,0) = \overline{v}(x) \ , \ \underline{z}(x,T) = \underline{v}(x) \ , \ x \in [0,a].$$

Next theorem is analogous to 3.III; it follows from (5.I) and the above remark in essentially the same way in which 3.III follows from 3.II; its proof is omitted since it is very similar to that of 3.III.

5.III THEOREM Put $\mathcal{R} = \mathcal{R}(1,\infty)$; fix $m = \overline{m} + \underline{m}$, $(A,f) \in \tilde{\Sigma}(\mathcal{R},m,\alpha)$ and $\overline{u} \in C^1([0,\infty],R_{\overline{m}})$ with bounded support.

Conclusion: there are $c_o > 0$, $c > 0$ real such that if $\phi \in C^1([0,1],R_m)$, $\phi\ \overline{u}$ satisfy (5.5) (i), $\max(|\phi|,|u|) \leq c_o$ and $\max(|\phi'|,|\overline{u}'|) \leq c$, then there exist $0 < T < \infty$ and $\underline{u} \in C^1([0,T],R_{\underline{m}})$ so that the solution $z = z(x,t)$ of (5.2) to (5.4) with $a = 1$ exists in $C^1(\mathcal{R}(1,T),R_m)$, is unique there and moreover satisfies

$$z(x,T) = 0 \quad \text{all} \quad x \in [0,1] \ ; \ |z| \leq 2c_o .$$

By taking $\overline{u} = 0$ in 5.III one obtains

5.IV COROLLARY Suppose $\mathcal{R} = \mathcal{R}(1,\infty)$, $m = \overline{m} + \underline{m}$ and $(A,f) \in \tilde{\Sigma}(\mathcal{R},m,\alpha)$. Then $z_t + A(x,t,z)z_x = f(t,z)$ is zero controllable with one boundary control.

Next theorem follows from 5.I and 5.III; its proof is omitted because it can be obtained by making minor modifications in that of 4.IV

5.V THEOREM Suppose $\mathcal{R} = \mathcal{R}(1,\infty)$, $m = \overline{m} + \underline{m}$ and $(A,f) \in \tilde{\Sigma}(\mathcal{R},m,\alpha)$. Conclusion: there are $c_o > 0$, $c > 0$ real such that if $\phi \in \Omega(c_o, c)$ there are $0 < T < \infty$, $\overline{u} \in C^1([0,T], R_{\overline{m}})$, $\underline{u} \in C^1([0,T], R_{\underline{m}})$ so that the solution $z = z(x,t)$ of (5.2) to (5.4) with $a = 1$ exists in $C^1(\mathcal{R}(1,T), R_m)$ and satisfies $z(x,t) = 0$ all $x \in [0,1]$.

Thus $z_t + A(x,t,z)z_x = f(t,z)$ is zero controllable with two boundary controls.

6. EXAMPLE: THE WAVE EQUATION

Consider the following boundary control problem for a nonlinear wave equation: to find $T > 0$ and real valued functions $\underline{u} = \underline{u}(t)$ on $[0,T]$, $w = w(x,t)$ on $\mathcal{R} \doteq [0,1] \times [0,T]$ such that

$$(1) \qquad w_{tt} = g^2(u_x)w_{xx} \;, \quad (x,t) \in \mathcal{R}$$

$$(2) \qquad w(x,0) = f(x) \;, \quad w_t(x,0) = h(x) \;, \quad x \in [0,1]$$

$$(3) \qquad w(0,t) = 0 \;, \quad w_x(1,t) = \underline{u}(t) \;, \quad t \in [0,T]$$

$$(4) \qquad w(x,T) = 0 \;, \quad x \in [0,1]$$

where g, f, h are given real valued functions of real variable and f,h satisfy appropriate compatibility conditions at $x = 0$.

If g is specialized to

$$(5) \qquad g(q) = \left(1 + E\left(1 - \frac{1}{\sqrt{1 + q^2}}\right)\right)^{\frac{1}{2}} \;, \quad q \in R$$

where $E > 0$ is a certain constant (Young's modulus), it is shown in [6] chapter 3 that a function w satisfying (1) describes the transverse planar vibration of an elastic string. If, moreover, $\underline{u}$ is given then the initial value problem (1) to (3) can be thought of approximating the transverse planar vibration of a string with given initial state f,h, one end clamped at $x = 0$, and the other end free to move at $x = 1$, along the straight line orthogonal to the x axis contained in the plane of motion, and subject to the external action $\overline{u} = \overline{u}(t)$.

If $z \in R_2$, let $\overline{z}$, $\underline{z}$ be respectively the first and the second component of z; it is easily seen that the transformation

$$\overline{z} = w_t \ , \ \underline{z} = w_x$$

reduces (1)... (4) to

$$z_t + A(z)z_x = 0 \ , \ (x,t) \in \mathcal{R}$$

$$z(x,0) = \phi(x) \ , \ x \in [0,1]$$

$$\overline{z}(0,t) = 0 \ , \ \underline{z}(1,t) = \underline{u}(t) \ , \ t \in [0,T]$$

$$z(x,T) = 0 \ , \ x \in [0,1].$$

where

$$A(z) = \begin{pmatrix} 0 & -g(\overline{z}) \\ -1 & 0 \end{pmatrix} \ , \ \overline{\phi} = f' \ , \ \underline{\phi} = h.$$

Also the eigenvalues of $A(z)$ are $\pm g(\overline{z})$; so if

(6) $\quad 0 < a < \infty, \ g \in C^1([-a,a],R) \ , \ g(0) \neq 0$

then A satisfies all the hypotheses of corollary 4.I, therefore as a particular case one obtains

PROPOSITION Suppose g satisfies (6). Then the wave equation (1) is zero controllable with one boundary control.

Thus whenever f and h are conveniently restricted there are T, $\underline{u}$, w satisfying (1) to (4). It is clear that if (6) holds then (1) is also controllable with two boundary controls.

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