23

APPLICATION OF LINEAR SPLINE FUNCTIONS TO THE NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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1. Introduction

This paper considers the use of linear spline functions to obtain an approximate numerical solution of the Volterra integral equation of the second kind,

$$(1) \qquad y(x) = \int_0^x K(x, s, y(s))\, ds + f(x) , \qquad x \geq 0 ,$$

where $y(x)$ is the unknown function, and the kernel $K(x, s, y(s))$ and $f(x)$ are given. We make the following assumptions:

(a) $f(x)$ is continuous and bounded on $0 \leq x \leq b$,

(b) $K(x, s, y)$ is uniformly continuous in x and s for all finite y and $0 \leq s \leq x \leq b$,

(c) $K(x, s, y)$ satisfies a uniform Lipschitz condition

$$|K(x, s, y_1) - K(x, s, y_2)| \leq L\, |y_1 - y_2|$$

for all $0 \leq s \leq x \leq b$. L is a constant independent of x and s .

These conditions guarantee the existence of a unique continuous solution to (1) (see [1]).

The justification for describing this simple method in such detail is similar to the justification for dealing with Euler's method for ordinary differential equation before describing more sophisticated methods — this simple case exemplifies some of the important features of this type of method without obscuring the analysis with the complication that arises in higher order methods.

The method is described in Section 2. In Section 3 it is shown that the method is convergent. An asymptotic formula for the discretization error is obtained in Section 4. The effects of the rounding error are analyzed in Section 5. Some numerical results are presented in Section 6.

2. Description of the method

Let $x_i = ih$, $i = 0, 1, 2, \ldots$ where h is an arbitrary constant step length. Let y_i denotes an approximation to $y(x_i)$, the exact value of $y(x)$ at $x = x_i$. We use a linear spline function $p(x)$, with knots at the points x_i, as an approximation to $y(x)$, i.e., for $i = 0, 1, \ldots$

$$(2) \qquad p(x) = \frac{1}{h} \left[(x_{i+1} - x)\, y_i + (x - x_i)\, y_{i+1} \right], \qquad x_i \leq x \leq x_{i+1} .$$

The function $p(x)$ is continuous at the knots.

The approximate solution of the integral equation is obtained by requiring that (1) be satisfied at the knots x_i, i.e., the exact solution $y(x)$ is replaced by the approximate solution $p(x)$ derived from values $p(x_i) = y_i$ computed from:

$$(3) \qquad p(x_{k+1}) = \int_0^{x_{k+1}} K(x_{k+1}, s, p(s))\, ds + f(x_{k+1}) \; .$$

This can be rewritten in the form:

$$(4) \qquad y_{k+1} = \int_0^{x_{k+1}} K(x_{k+1}, s, \frac{1}{h}[(x_{k+1} - s)\, y_k + (s - x_k)\, y_{k+1}])\, ds + r_{k+1} \; ,$$

where r_{k+1} does not depend on y_{k+1} :

$$r_{k+1} = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(x_{k+1}, s, \frac{1}{h}[(x_{i+1} - s)\, y_i + (s - x_i) y_{i+1}])ds + f(x_{k+1}).$$

Equation (4) must be solved for y_{k+1}. Consider

$$F(z) = \int_{x_k}^{x_{k+1}} K(x_{k+1}, s, \frac{1}{h}[(x_{k+1} - s)\, y_k + (s - x_k) z])\, ds + r_{k+1} \; .$$

It is easy to show, using the Lipschitz condition (c) on K , that

$$(5) \qquad |F(z) - F(u)| \leq \frac{1}{2} L\, h\, |z - u| \; .$$

The equation (4) is

$$y_{k+1} = F(y_{k+1}) \; .$$

Equation (5) shows that F is a strong contraction mapping for $h < 2/L$, so that, if this condition is satisfied, (4) has a unique fixed point y_{k+1}, that may be found by iteration.

Since $y(0) = f(0)$, we can take $y_0 = f(0)$ as the initial condition. The values of $y_1, y_2, \ldots$ can then be determined successively from (4).

An estimate of $y'(x)$ is given by the derivative of (2). If we denote this (constant) estimate of $y'(x)$ in $x_k \le x < x_{k+1}$ by y'_k, this gives

(6) $$y'_k = \frac{1}{h}(y_{k+1} - y_k), \qquad x_k \le x < x_{k+1} .$$

If the integral equation is linear, say $K(x, s, y(s)) = k(x, s)\, y(s)$, then (4) can be rearranged to give y_{k+1} explicitly:

(7) $$y_{k+1} = \frac{\alpha y_k + r_{k+1}}{1 - \beta} ,$$

where

$$\alpha = \frac{1}{h}\int_{x_k}^{x_{k+1}} k(x_{k+1}, s)(s_{k+1} - s)\, ds \quad , \quad \beta = \frac{1}{h}\int_{x_k}^{x_{k+1}} k(x_{k+1}, s)(s - x_k)\, ds .$$

3. <u>An a priori bound and convergence</u>

The proofs for Theorem 1, 3, and 5 require the following lemma.

Lemma 1. If $|Z_k| \le A \sum_{i=0}^{k-1} |Z_i| + B$ for $k = 1, 2, \ldots$ with $A > 0$, $B > 0$, and $|Z_0| \le C$, then $|Z_k| \le (B + AC)(1 + A)^{k-1}$ for $k = 1, 2, \ldots$.

The proof of this lemma can be found in [2], p. 7. Note that if $A = hN$, and $kh = x$, then

$$|Z_k| \le (B + hNC) \exp(Nx) .$$

Let $y(x)$ be the exact solution of (1), and define the discretization error function $E(x)$ by

$$E(x) = y(x) - p(x) ,$$

where $p(x)$ is the spline function approximation to $y(x)$ obtained from our numerical method. Assume that $y(x)$ is twice continuously differentiable on $[0, b]$. Since $p(x)$ is a linear spline, we have $p''(x) = 0$, $x_i < x < x_{i+1}$, so that, by Taylor's expansion

$$(8) \qquad E(x) = E(x_i) + (x - x_i)\, E'(x_i) + \frac{1}{2}(x - x_i)^2\, y''(\xi_x), \qquad x_i < \xi_x < x,$$

$$(9) \qquad E'(x) = E'(x_i) + (x - x_i)\, y''(\eta_x), \qquad x_i < \eta_x < x,$$

where $x \in [x_i, x_{i+1})$. Denote the value of ξ_x, when $x = x_{i+1}$, by ξ. Then if we set $x = x_{i+1}$ in equation (8) we obtain

$$(10) \qquad E'(x_i) = \frac{E(x_{i+1}) - E(x_i)}{h} - \frac{1}{2}\, h\, y''(\xi), \qquad x_i < \xi < x_{i+1}.$$

Substituting (10) into (8), we obtain

$$(11) \qquad E(x) = \frac{1}{h}\left[E(x_i)(x_{i+1} - x) + E(x_{i+1})(x - x_i)\right] + \left[y''(\xi_x)(x - x_i) - y''(\xi)h\right] \frac{(x - x_i)}{2},$$

for $x \in [x_i, x_{i+1})$.

If we define $N(x) = \max_{t \in [0, x]} |y''(t)|$, then for $x \in [x_i, x_{i+1})$

$$(12) \qquad |E(x)| \le |E(x_i)| + |E(x_{i+1})| + N(x_{i+1})\, h^2.$$

Since both $y(x)$ and $p(x)$ satisfy (1) at $x = x_k$, $k = 1, 2, \ldots,$ therefore

$$(13) \qquad E(x_k) = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \left[K(x_k, s, y(s)) - K(x_k, s, p(s))\right] ds,$$

which implies

$$(14) \qquad |E(x_k)| \le \sum_{i=0}^{k-1} L\,h\,|E(x)| \ .$$

By means of (12), (14) can be rewritten as

$$(15) \qquad |E(x_k)| \le L\,h \sum_{i=0}^{k-1} \left(|E(x_{i+1})| + |E(x_i)| + N(x_{i+1})\,h^2 \right) .$$

Transferring $L h\,|E(x_k)|$ from the right to the left of the inequality, it is permissible to divide by $1 - Lh$ if $Lh < 1$, and this gives

$$(16) \qquad |E(x_k)| \le \frac{2\,L\,h}{1 - Lh} \sum_{i=0}^{k-1} |E(x_i)| + \frac{L\,k\,h^3}{1 - Lh} N(x_k) \ ,$$

with $E(x_0) = 0$. A bound on $|E(x_k)|$ can be obtained by Lemma 1 . We formulate the result in the following theorem:

Theorem 1. Let $K(x, s, y)$ and $f(x)$ satisfy conditions (a), (b), (c), and let the exact solution $y(x) \in C^2[0, b]$. If

$$N(x) = \max_{t \in [0, x]} |y''(t)| \ ,$$

then the discretization error of the method satisfies

$$(17) \qquad |E(x_k)| \le \left(\frac{L\,h^2}{1 - Lh} N(x_k)\,x_k \right) \exp\left(\frac{2\,L}{1 - Lh} \right) x_k \ ,$$

for $k = 0, 1, 2, \ldots,$ provided $h < \frac{1}{L}$.

It is obvious from equation (17) that our method is exact for any Volterra equation of the second kind whose solution is linear in x .

Corollary 1.1 If the assumptions of Theorem 1 are satisfied, then there exists a constant C such that for $x \in [0, b]$

$$|E(x)| < C h^2 ,$$

$$|E'(x) < C h .$$

Proof: Since equation (17) implies that

$$(18) \qquad |E(x_i)| = 0(h^2) ,$$

therefore from (10) we have

$$(19) \qquad |E'(x_i)| = 0 (h) .$$

Corollary 1.1 immediately follows from (8) and (9) by using (18) and (19).

In Corollary 1.1 we have shown that the error of approximating the derivative is of order h for all $x \in [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots$. In the followings we can show, with additional assumptions, that the error is of order h^2 at the mid-point of each interval.

We assume that $y \in C^3[0, b]$, so that for $x \in [x_i, x_{i+1})$, $E(x)$, $E'(x)$ can be expanded respectively in Taylor's series about $x = x_i$ to one more term than in (8) and (9):

$$(20) \qquad E(x) = E(x_i) + (x - x_i) E'(x_i) + \frac{1}{2} (x - x_i)^2 y''(x_i) + \frac{1}{6} (x - x_i)^3 y'''(\xi_x), x_i < \xi_x < x,$$

$$(21) \qquad E'(x) = E'(x_i) + (x - x_i)y''(x_i) + \frac{1}{2} (x - x_i)^2 y'''(\eta_x) , \qquad x_i < \eta_x < x .$$

Putting $x = x_{i+1}$ in (20), solving the resulting equation for $E'(x_i)$, and substituting the result in (20) gives

(22) $$E(x) = \frac{1}{h}\left[(x_{i+1} - x)E(x_i) + (x - x_i)E(x_{i+1})\right] + \varphi(x) ,$$

where

(23) $$\varphi(x) = (x - x_i)(x - x_{i+1})\frac{y''(x_i)}{2} + \left[(x - x_i)^2 y'''(\xi_x) - h^2 y'''(\xi)\right]\frac{(x - x_i)}{6} ,$$

$$x_i < \xi < x_{i+1} ,$$

and ξ, which appears in (23), denotes the value of ξ_x at $x = x_{i+1}$.

Assume that $K(x, s, y)$ not only satisfies conditions (b) and (c), but also has continuous and bounded first and second direvatives with respect to y in $0 \le s \le x \le b$. Under this hypothesis we may write, by Taylor's formula, that

(24) $$K(x_j, s, y(s)) - K(x_j, s, p(s)) = K_y(x_j, s, y(s))E(s) + \frac{1}{2}K_{yy}(x_j, s, y^*(s))E^2(s),$$

where $y^*(s)$ is between $y(s)$ and $p(s)$.

By using Corollary 1.1, we have from (24) that

(25) $$K(x_j, s, y(s)) - K(x_j, s, p(s)) = K_y(x_j, s, y(s))E(s) + 0(h^4) .$$

Letting $k = j$ in equation (13), and substituting (25) into the resulting equation we have

(26) $$E(x_j) = \sum_{i=0}^{j-1} \int_{x_i}^{x_{i+1}} K_y(x_j, s, y(s))E(s)\,ds + 0(h^4) .$$

Taking the difference of the two equations resulting from replacing j by k and $k+1$ in (26) gives us

$$(27) \qquad E(x_{k+1}) - E(x_k) = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} [K_y(x_{k+1}, s, y) - K_y(x_k, s, y)] \, E(s)\, ds$$

$$+ \int_{x_k}^{x_{k+1}} K_y(x_{k+1}, s, y)\, E(s)\, ds + 0(h^4).$$

If we assume that $K_y(x, s, y)$ satisfies the Lipschitz condition

$$(28) \qquad |K_y(x^*, s, y) - K_y(\overline{x}, s, y)| \le L_1 \, |x^* - \overline{x}| , \qquad 0 \le x^*, \overline{x} \le b ,$$

for all finite y, and all $0 \le s \le b$, then from equation (27) we can conclude that

$$(29) \qquad |E(x_{k+1}) - E(x_k)| \le L_1 h^2 \sum_{j=0}^{k-1} |E(s)| + 0(h^3) ,$$

which implies, by using Corollary 1.1, that

$$(30) \qquad |E(x_{k+1}) - E(x_k)| = 0(h^3)$$

Taking the derivative of equation (22) and letting $i = k$ we have

$$(31) \qquad E'(x) = \frac{1}{h}(E(x_{k+1}) - E(x_k)) + y''(x_k)(x - x_k - \frac{h}{2}) + 0(h^2) .$$

Putting $x = x_k + \frac{h}{2}$ in the resulting equation, and using (30) we can conclude that

$$(32) \qquad |E'(x_k + \frac{h}{2})| = 0(h^2)$$

We formulate the result into the following theorem:

Theorem 2. Let the assumptions of Theorem 1 be satisfied. Assume that $y \in c^3[0,b]$, and $K(x,s,y)$ has continuous and bounded first and second derivatives with respect to y. If $K_y(x,s,y)$ satisfies the Lipschitz condition (28), then there exists a constant c such that for $k = 0, 1, 2, \ldots.$

$$(33) \qquad |y'(x_k + \frac{h}{2}) - p'(x_k + \frac{h}{2})| < ch^2 .$$

Corollary 2.1. Let the assumptions of Theorem 2 be satisfied. If we define

$$(34) \qquad \bar{p}'(x_k) = \frac{1}{2}[p'(x_k - \frac{h}{2}) + p'(x_k + \frac{h}{2})] , \qquad k = 1, 2, \ldots,$$

then there exists a constant c such that

$$(35) \qquad |y'(x_k) - \bar{p}'(x_k)| < ch^2 .$$

Proof: By (33) and (34),

$$\bar{p}'(x_k) = \frac{1}{2}[p'(x_k - \frac{h}{2}) + p'(x_k + \frac{h}{2})]$$

$$= \frac{1}{2}[y'(x_k - \frac{h}{2}) + y'(x_k + \frac{h}{2})] + 0(h^2) .$$

But since $y \in c^3[0,b]$,

$$y'(x_k - \frac{h}{2}) = y'(x_k) - \frac{1}{2}h\,y''(x_k) + \frac{1}{8}h^2\,y'''(\xi_1) , \qquad x_k - \frac{h}{2} < \xi_1 < x_k ,$$

$$y'(x_k + \frac{h}{2}) = y'(x_k) + \frac{1}{2}h\,y''(x_k) + \frac{1}{8}h^2\,y'''(\xi_2) , \qquad x_k < \xi_2 < x_k + \frac{h}{2} .$$

Thus we finally obtain

(36) $\quad \overline{p}'(x_k) = y'(x_k) + 0(h^2)$.

This completes the proof.

Corollary 2.1 states that we can obtain a much improved estimate of the derivative at the knots by setting this equal to the mean of the slopes of the linear approximation function in the two intervals on either side of the knots.

We conclude this section with a result which is similar in content to Theorem 1, but is valid under somewhat relaxed conditions. We use this theorem in Section 4 when finding the asymptotic formula for the discretization error.

Theorem 3. Let the assumptions of Theorem 1 be satisfied, and let $p(x)$ be the linear spline as defined by (2). If $\{y_k\}$ be the sequence of numbers generated from

$$(37) \qquad y_k = \int_{x_{k-1}}^{x_k} K(x_k, s, p(s))\, ds + r_k + \theta_k h^2 c, \qquad k = 1, 2, \ldots,$$

$$y_0 = f(0) \; ,$$

where $c \geq 0$ is a constant, r_k as defined in (4), and the θ_k are numbers which may vary from step to step, but always satisfy $|\theta_k| \leq 1$. Then for $k = 0, 1, 2, \ldots$

$$(38) \qquad |E(x_k)| \leq h^2 \left(\frac{L\, N(x_k)x_k + c}{1 - L h}\right) \exp\left(\frac{2\,L}{1 - Lh}\right) x_k \; ,$$

where $E(x_k) = y(x_k) - y_k$.

The point of Theorem 3 consists in showing that even if the recurrence relation (4) are not satisfied exactly, the values y_k may still converge to $y(x_k)$ provided that the discrepancy between (4) and (37) is not too great.

The proof of Theorem 3 proceeds in the same way as did the proof of Theorem 1. In place of (15), we now obtain

$$(39) \qquad |E(x_k)| \leq L\,h \sum_{i=0}^{k-1} \left(|E(x_{i+1})| + |E(x_i)| + N(x_{i+1})h^2\right) + ch^2 ,$$

which implies

$$(40) \qquad |E(x_k)| \leq \frac{2\,L\,h}{1-Lh} \sum_{i=0}^{k-1} |E(x_i)| + \left(\frac{L\,k\,h^3\,N(x_k) + ch^2}{1 - L\,h}\right) .$$

Using Lemma 1, we then have (38) .

4. An asymptotic formula for the discretization error

The error bounds derived in Section 3 generally overestimate the actual error by a considerable amount. In this section we obtain an asymptotic formula for the error, which leads to error estimates.

Let us now return to equation (26) in Section 3, and let $j = k + 1$ in that equation, then we have

$$(41) \qquad E(x_{k+1}) = \sum_{i=0}^{k} \int_{x_i}^{x_{i+1}} K_y(x_{k+1}, s, y(s))\, ds + 0(h^4) .$$

We divide the resulting relation by h^2 , and introduce the quantities $\overline{E}(x_i) = h^{-2} E(x_i)$. By means of (22), equation (41) can be rewritten in the form:

(42) $$\bar{E}(x_{k+1}) = \frac{\gamma\ \bar{E}(x_k) + S_{k+1}}{1 - \delta} ,$$

where

$$\gamma = \frac{1}{h}\int_{x_k}^{x_{k+1}} K_y(x_{k+1}, s, y)(x_{k+1} - s)\, ds, \quad \delta = \frac{1}{h}\int_{x_k}^{x_{k+1}} K_y(x_{k+1}, s, y)(s - x_k) ds ,$$

and

$$S_{k+1} = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K_y(x_{k+1}, s, y)\ \frac{(x_{i+1} - s)\bar{E}(x_i) + (s - x_i)\bar{E}(x_{i+1})}{h}\, ds + g(x_{k+1}) + 0(h^2),$$

with $g(x_{k+1}) = \sum_{i=0}^{k} \int_{x_i}^{x_{i+1}} K_y(x_{k+1}, s, y)\ \varphi(s) h^{-2}\, ds$,

where $\varphi(s)$ as defined by (23).

Defining $\bar{K}(x, s) = K_y(x, s, y(s))$, and comparing (42) with (7), we can look at (42) as the result of applying the linear spline method to the solution of a new integral equation for a function $e(x)$:

(43) $$e(x) = \int_0^x \bar{K}(x, s)\ e(s)\, ds + \psi(x) ,$$

where

$$\psi(x) = \int_0^x \bar{K}(x, s)\ \varphi(s)\, h^{-2}\, ds$$

making at each step an additional error of order h^2 .

Note that for sufficiently small h , the second and third therm of $\varphi(s)$ can be neglected, and we can approximate $\psi(x)$ by

$$\psi(x) \simeq -\frac{1}{12}\int_0^x \bar{K}(x, s)\ y''(s)\, ds .$$

$E(x_0) = 0$ implies that $\overline{E}(x_0) = 0$. To equation (43) we can apply Theorem 3 with the following result:

Theorem 4. Let the assumptions of Theorem 2 be satisfied. Then the error $E(x_k)$ of the linear spline approximation to the solution $y(x)$ of equation (1) can be written in the form:

$$(44) \qquad E(x_k) = h^2 e(x_k) + 0(h^3) , \qquad k = 1, 2, \ldots,$$

where $e(x)$ is the solution of

$$(45) \qquad e(x) = \int_0^x K_y(x, s, y(s))\, e(s)\, ds + \psi(x) + 0(h) ,$$

where

$$\psi(x) = -\frac{1}{12} \int_0^x K_y(x, s, y(s))\, y''(s)\, ds ,$$

provided that h is sufficiently small.

Corollary 4.1. If the assumptions of Theorem 4 are satisfied then for $x \in [x_k, x_{k+1})$, the discretization error function $E(x)$ along with its first derivative can be written in the following form:

$$(46) \qquad E(x) = h^2 e(x_k) + \frac{y''(x_k)}{2} (x - x_k)(x - x_{k+1}) + 0(h^3) ,$$

$$(47) \qquad E'(x) = y''(x_k)\,(x - x_k - \frac{h}{2}) + 0(h^2) ,$$

where $e(x)$ is the solution of (45) .

Proof: Setting $x = x_k$, and using (30), we can have from equation (31) that

$$(48) \qquad E'(x_k) = -\frac{1}{2} y''(x_k)h + 0(h^2) \ .$$

By means of (44) and (48) we can obtain (46) and (47) respectively from (20) and (21) .

4. The round off error of the method.

We have dealt with the behavior of errors under the assumption that the numerical solutions y_k, y'_k strictly satisfy (4) and (6). In actual computation, however, y_k and y'_k do not satisfy these equations, because of the effect of round off. The calculated value of the numerical solutions (denoted by $\overline{y}_k$, $\overline{y}'_k$) satisfy slightly perturbed equations:

$$(49) \qquad \overline{y}_k = \int_{x_{k-1}}^{x_k} K(x_k, s, \frac{1}{h}[(x_k - s)\,\overline{y}_{k-1} + (s - x_{k-1})\,\overline{y}_k])\,ds + \overline{r}_k + \xi_k ,$$

where

$$\overline{r}_k = \sum_{i=0}^{k-2} \int_{x_i}^{x_{i+1}} K(x_k, s, \frac{1}{h}[(x_{i+1} - s)\,\overline{y}_i + (s - x_i)\,\overline{y}_{i+1}])\,ds + f(x_k),$$

and

$$(50) \qquad \overline{y}'_k = \frac{1}{h}(\overline{y}_{k+1} - \overline{y}_k) + \eta_k \ .$$

Here ξ_k, η_k are the local round off error. We shall compare the "actual" numerical solutions $\overline{y}_k$, $\overline{y}'_k$ with the "theoretical" numerical solutions y_k, y'_k .

Define the accumulated round off errors in computing y_k, y'_k respectively by $R_k = \overline{y}_k - y_k$, $R'_k = \overline{y}'_k - y'_k$. Subtraction of equations (4) and (6) from (49) and (50), replacing $k+1$ by k in the former equation, gives

(51) $$R_k = \int_{x_{k-1}}^{x_k} \{K(x_k, s, \frac{1}{h}[(x_k - s)\,\overline{y}_{k-1} + (s - x_{k-1})\,\overline{y}_k]) - K(x_k, s, \frac{1}{h}[(x_k - s)\,y_{k-1} + (s - x_{k-1})y_k])\}\,ds + (\overline{r}_k - r_k) + \xi_k,$$

(52) $$R'_k = \frac{1}{h}(R_{k+1} - R_k) + \eta_k .$$

Using the Lipschitz condition (c) on equation (44), we have

(53) $$|R_k| \le L\,h \sum_{i=0}^{k-1} (|R_{i+1}| + |R_i|) + |\xi_k| ,$$

which implies

(54) $$|R_k| \le \frac{2\,L\,h}{1 - Lh} \sum_{i=0}^{k-1} |R_i| + \frac{|\xi_k|}{1 - Lh} , \qquad k = 1, 2, \ldots .$$

By using Lemma 1 on (54) we obtain

(55) $$|R_k| \le \left(\frac{|\xi_k| + 2\,L\,h\,|R_0|}{1 - L\,h}\right) \exp\left(\frac{2L}{1 - Lh}\right) x_k ,$$

where R_0 is the round off error in computing $f(0)$.

Using (55) we can conclude from (52) that

(56) $$|R'_k| \le \frac{2}{h}\left(\frac{|\xi_{k+1}| + 2\,L\,h\,|R_0|}{1 - L\,h}\right) \exp\left(\frac{2\,L}{1 - L\,h}\right) x_{k+1} + |\eta_k| .$$

We formulate the above result in the following theorem:

Theorem 5. If $K(x, s, y)$ and $f(x)$ satisfy conditions (a), (b), (c), then the accumulated round off R_k, R'_k in computing y_k, y'_k respectively from (4) and (6) obey (55) and (56) provided $h < \frac{1}{L}$.

Although the bound in this theorem may be unrealistic, the important feature is that $|R_k|$, the accumulated round off error, is bounded as $h \to 0$ for fixed x .

5. Numerical examples.

Numerical results are computed and tabulated for two examples. Example 1 is nonlinear, whereas Example 2 has a singular kernel. The numerical procedure used is exactly that described in Section 2. It is self-starting, and the step-size can be changed at any step, if necessary, without added complications.

In what follows, for the sake of simplicity, we shall refer to the point which is mid-way between two knots as the mid-point. In both of the examples in the tables, the step-size h is kept constant throughout the range of integration. The size of h is reduced by a factor of three each time in order that knots will remain as knots, and mid-points will remain as mid-points. With such an arrangement, the speed of convergence of the results at the points under consideration will become clear directly from the numerical results.

We tabulate, for each example, the results for only two points, one knot and one mid-point. These illustrate some of the features of the method. In Example 1, $x = 1.25$ is a mid-point and $x = 1.50$ is a knot, whereas in Example 2, $x = 0.25$ is a mid-point and $x = 0.5$ is a knot.

In Tables 1a and 2a , the column 5 and 8 illustrate the speed of convergence of the approximation $p(x)$ and its dirivate $p'(x)$:

(i) As h is reduced by a factor of 3, the error of $p(x)$ at the mid-point and at the knot is reduced by a factor of approximately 9. This checks with Theorem 1 and its corollary 1.1 which state that

$$y(x) - p(x) = 0(h^2)$$

for $x \in [0, b]$.

(ii) As h is reduced by a factor of 3, the error of $p'(x)$ at the mid-point is reduced by a factor of approximately 9. This verifies Theorem 2, which states that

$$y'(x_k + \frac{h}{2}) - p'(x_k + \frac{h}{2}) = 0(h^2)$$

for $k = 0, 1, 2, \ldots$.

(iii) The slope $p'(x)$ is constant in $x_k < x < x_{k+1}$. In Tables 1a, 2a we have taken $p'(x)$ at the knot x_k to be the same as $p'(x)$ in $x_k < x < x_{k+1}$. From the tables it is seen that as h is reduced by a factor of 3, the error $p'(x)$ at the knot is reduced by a factor of approximately 3. This checks with the fact that

$$y'(x_k) - p'(x_k) = 0(h)$$

for $k = 0, 1, 2, \ldots$.

To improve the estimate of the derivative at the knot we compute it by taking the mean of the slopes of the linear approximation function in the two intervals on either side of the knot. Referring to Tables 1b and 2b, it is seen that as h is reduced by a factor of 3, the error of the improved estimate is reduced by a factor of approximately 9. This verifies Corollary 2.1 which states that

$$y'(x_k) - \bar{p}'(x_k) = 0(h^2) , \qquad k = 1, 2, \ldots,$$

provided that we define

$$\bar{p}'(x_k) = \frac{1}{2} [p'(x_k - \frac{h}{2}) + p'(x_k + \frac{h}{2})] .$$

No signs of instability were present in the numerical results obtained by this method applied to the two examples.

Example 1. Numerical solution of

$$y(x) = \int_0^x y^2(s)\,ds + e^{-x} + \frac{1}{2}(e^{-2x} - 1) .$$

Exact solution is $y(x) = e^{-x}$.

Table 1a.

x	h	$y(x)$	$p(x)$	$y(x)-p(x)$	$y'(x)$	$p'(x)$	$y'(x)-1'(x)$
1.25	0.1	.286504	.288778	-2.27×10^{-3}	-.286504	-.285384	-1.12×10^{-3}
	$\frac{1}{3}(0.1)$		.286757	-2.53×10^{-4}		-.286380	-1.24×10^{-4}
	$\frac{1}{9}(0.1)$		.286533	-2.90×10^{-5}		-.286491	-1.37×10^{-5}
1.50	0.1	.223130	.225335	-2.21×10^{-3}	-.223130	-.211297	-1.18×10^{-2}
	$\frac{1}{3}(0.1)$		.223374	-2.44×10^{-4}		-.219335	-3.79×10^{-3}
	$\frac{1}{9}(0.1)$		.223157	-2.72×10^{-5}		-.221882	-1.25×10^{-3}

$y(x)$ = exact solution, $p(x)$ = approximation solution, h = step-size .

Table 1b.

h	$y'(1.5)$	$\bar{p}'(1.5)$	$y'(1.5)-\bar{p}'(1.5)$
0.1	-.223130	-.222430	-7.00×10^{-4}
$\frac{1}{3}(0.1)$		-.223053	-7.71×10^{-5}
$\frac{1}{9}(0.1)$		-.223122	-8.50×10^{-6}

$\bar{p}'(1.5)$ = improved estimate of the derivative at $x = 1.5$.

Example 2. Numerical solution of

$$y(x) = -\int_0^x \frac{1}{\sqrt{x-s}}\, y(s)\, ds + \frac{1}{1+x} + \frac{2}{\sqrt{1+x}} \log(\sqrt{1+x} + \sqrt{x}).$$

Exact solution is $y(x) = \dfrac{1}{1+x}$.

Table 2a

x	h	y(x)	p(x)	y(x)-p(x)	y'(x)	p'(x)	y'(x) - p'(x)
0.25	0.1	.800000	.800770	-7.70×10^{-4}	-.640000	-.640712	7.12×10^{-4}
	$\frac{1}{3}(0.1)$		.800084	-8.44×10^{-5}		-.640074	7.42×10^{-5}
	$\frac{1}{9}(0.1)$		.800009	-9.26×10^{-6}		-.640007	7.70×10^{-6}
0.50	0.1	.666666	.666261	4.05×10^{-4}	-.444444	-.416241	2.82×10^{-2}
	$\frac{1}{3}(0.1)$		.666621	4.52×10^{-5}		-.434732	-9.71×10^{-3}
	$\frac{1}{9}(0.1)$		.666661	5.04×10^{-6}		-.441170	-3.27×10^{-3}

y(x) = exact solution, p(x) = approximation solution, h = step-size .

Table 2b

h	y'(.5)	$\bar{p}'(.5)$	$y'(.5)-\bar{p}'(.5)$
0.1	-.444444	-.445987	1.54×10^{-3}
$\frac{1}{3}(0.1)$		-.444612	1.68×10^{-4}
$\frac{1}{9}(0.1)$		-.444463	1.85×10^{-5}

$\bar{p}'(.5)$ = improved estimate of the derivative at x = 0.5 .

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References

[1] Tricomi, F. G., Integral Equations, Interscience, New York (1957).

[2] Linz, P., The numerical solution of Volterra integral equations by finite difference methods, Mathematics Research Center Technical Summary Report #825, November 1967.