19

CONTINUITY IN THE STRONG TOPOLOGY OF OPERATOR VALUED SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH AN APPLICATION TO OPTIMAL CONTROL

by

David L. Russell

Computer Sciences Technical Report #23

May 1968

Continuity in the Strong Topology of Operator Valued Solutions of Nonlinear Differential Equations with an Application to Optimal Control*

by

David L. Russell**

0. Introduction.

A classical result in the theory of ordinary differential equations concerns continuity of solutions with respect to initial conditions and parameters. If for $x \in R^n$, $\mu \in R^m$, t real, we denote by $x(t, \xi, \mu)$ the solution of

$$(0.1) \qquad \dot{x} = f(x, t, \mu)$$

which satisfies the initial condition

$$(0.2) \qquad x(0, \xi, \mu) = \xi$$

then the familiar result is the following. Suppose that the solution $x(t, \xi_0, \mu_0)$ exists for $t \in [0, T)$. Given $\delta > 0$, if we choose $\|\xi - \xi_0\|$ and $\|\mu - \mu_0\|$ sufficiently small, then the solution $x(t, \xi, \mu)$ exists for $t \in [0, T - \delta]$. Moreover, uniformly for all $t \in [0, T - \delta]$

$$(0.3) \qquad \lim_{\substack{\xi \to \xi_0 \\ \mu \to \mu_0}} x(t, \xi, \mu) = x(t, \xi_0, \mu_0) .$$

Of course, certain rather general conditions must be imposed upon $f(x, t, \mu)$. These conditions as well as a proof of the result just indicated may be found, e.g., in [1] .

---

*Supported in part by the National Science Foundation under Grant GP 6070.

**Departments of Mathematics and Computer Sciences, Univ. of Wis., Madison.

In this paper we wish to consider differential equations with solutions $X(t)$, where $X(t) \in \beta(B_1, B_2)$, the Banach space of all bounded linear transformations $X: B_1 \to B_2$, where B_1 and B_2 are themselves Banach spaces. Now the usual topology used in $\beta = \beta(B_1, B_2)$ is the one induced by the norm

$$(0.4) \qquad \|X\| = \sup_{\substack{y \in B_1 \\ \|y\|_1 = 1}} \|Xy\|_2 ,$$

where $\|\cdot\|_j$ is the norm in B_j. As long as we speak of continuity with respect to this norm topology there is very little change from the results described above as we pass from finite dimensional systems to systems having operator valued solutions $X(t)$. There is, however, another topology in β which is of frequent interest. We say that a sequence $\{X_n\} \in \beta$ converges to $X_\infty \in \beta$ in the strong topology of β if for every $y \in B_1$

$$(0.5) \qquad \lim_{n \to \infty} \|X_n y - X_\infty y\|_2 = 0 .$$

We may then ask: given a differential equation with solutions in β, does continuity with respect to initial conditions and parameters prevail if we work only within the framework of the strong topology?

It is immediately clear that if any such results are obtained they must differ somewhat from those already known for (0.1). For example, let β denote the space of bounded linear transformations from the Hilbert space ℓ^2 into itself. An element $X \in \beta$ can be represented by an infinite matrix

$$(0.6) \qquad X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} .$$

Let us denote those X represented by diagonal matrices as

$$(0.7) \qquad X = \operatorname{diag}\,(x_{11}, x_{22}, x_{33}, \ldots) .$$

Consider then the differential equation

$$(0.8) \qquad \dot{X} = X^2$$

in $\beta = \beta(\ell^2, \ell^2)$. We prescribe initial conditions

$$(0.9) \qquad X_n(0) = \operatorname{diag}\,(\underbrace{-1, -1, \ldots, -1,}_{n \text{ entries}} 1, 1, 1, \ldots), \; n = 1, 2, 3, \ldots,$$

$$X_\infty(0) = \operatorname{diag}\,(-1, -1, -1, \ldots) .$$

It is clear that $\lim_{n \to \infty} X_n(0) = X_\infty(0)$ in the strong topology of β. Now the corresponding solutions of (0.8) are

$$(0.10) \qquad X_n(t) = \operatorname{diag}\,(\underbrace{\frac{1}{-1-t}, \ldots, \frac{1}{-1-t},}_{n \text{ entries}} \frac{1}{1-t}, \frac{1}{1-t}, \ldots),$$

$$X_\infty(t) = \operatorname{diag}\,(\frac{1}{-1-t}, \frac{1}{-1-t}, \frac{1}{-1-t}, \ldots) .$$

The solution $X_\infty(t)$ exists for all $t \geq 0$ while none of the solutions $X_n(t)$ is defined for $t \geq 1$. This is a definite departure from our experience with finite dimensional systems. Observe, however, that for all $t > 1$, $X_n(t)$ converges strongly to $X_\infty(t)$ as $n \to \infty$.

In section 1 we will study, under fairly general assumptions, the question of strong continuity of solutions with respect to initial conditions and parameters. Because we wish to include the case of "mild solutions" (cf. [2]) of certain differential equations whose linear part involves an unbounded operator, we work with the integral equation (1.6) below. This equation is of a type general enough to include most differential equations in β which are likely to be of interest.

In section 2 we will show that our results are of more than purely mathematical interest in that they can be applied to the study of finite dimensional approximations to certain optimization problems for differential equations in Hilbert spaces, or, in engineering parlance, distributed parameter systems.

1. Strong Continuity Results for Integral Equations

Let $\beta = \beta(B_1, B_2)$ be the Banach space of bounded linear transformations $X: B_1 \to B_2$ equipped with the norm (0.4) and let Ω be a compact topological space. For $X \in \beta$, $\mu \in \Omega$, t, s real, $0 \le s \le t$, let $F(X, \mu, t, s)$ and $G(X, \mu, t)$ be functions

$$F: \beta \otimes \Omega \otimes R^2 \to \beta \quad , \tag{1.1}$$

$$G: \beta \otimes \Omega \otimes R^1 \to \beta, \quad G(X, \mu, 0) \equiv X, \tag{1.2}$$

with the following properties.

Property (i) Let $X(\mu, t)$ and $X_0(\mu)$ be continuous relative to the strong topology of β for $\mu \in \Omega$, t real. Then $F(X(\mu, s), \mu, t, s)$ and $G(X_0(\mu), \mu, t)$ are both continuous relative to the strong topology of β for $\mu \in \Omega$, t and s real, $0 \le s \le t$.

Property (ii) Corresponding to any set

$$(1.3) \qquad 0 \le t \le T, \quad \|X\| \le K$$

there is a positive number $L = L(T, K)$ such that if X_1, X_2, t satisfy (1.3) then, uniformly for $\mu \in \Omega$, $0 \le s \le t \le T$,

$$(1.4) \qquad \|F(X_2, \mu, t, s) - F(X_1, \mu, t, s)\| \le L \|X_2 - X_1\|,$$

$$(1.5) \qquad \|G(X_2, \mu, t) - G(X_1, \mu, t)\| \le L \|X_2 - X_1\|.$$

Theorem 1. Let $X(\mu, t)$ be the solution of the integral equation

$$(1.6) \qquad X(\mu, t) = G(X_0(\mu), \mu, t) + \int_0^t F(X(\mu, s), \mu, t, s)\, ds$$

satisfying the initial condition

$$(1.7) \qquad X(\mu, 0) = X_0(\mu).$$

If $X_0(\mu)$ is strongly continuous for $\mu \in \Omega$ there is a positive number T such that $X(\mu, t)$ is strongly continuous in the set: $\mu \in \Omega$, $t \in [0, T]$.

Remark. The integral equation (1.6) clearly covers the case of a differential equation

$$(1.8) \qquad \dot{X} = P(X, \mu, t)$$

where $P(X, \mu, t)$ is a polynomial in X with coefficient operators which are strongly continuous functions for $\mu \in \Omega$, $t \in R^1$. This is true because multiplication of bounded operators preserves strong continuity.

Proof of Theorem 1. The usual proof of the local existence and uniqueness of solutions of (1.6) employs the method of successive approximations. One sets

$$X_0(\mu, t) \equiv X_0(\mu), \quad 0 \le t \le T, \tag{1.9}$$

and thereafter

$$X_{n+1}(\mu, t) = G(X_0(\mu), \mu, t) + \int_0^t F(X_n(\mu, s), \mu, t, s)\, ds . \tag{1.10}$$

Using properties (i) and (ii) above, one shows in a way which is by now familiar to all, that if T is sufficiently small

$$\lim_{n \to \infty} \| X_n(\mu, t) - X(\mu, t) \| = 0 \tag{1.11}$$

uniformly for $0 \le t \le T$, where $X(\mu, t)$ is the unique solution of (1.6) and (1.7).

For each fixed $y \in B_1$, $X_0(\mu)y \colon \Omega \to B_2$ is continuous with respect to the $\|\cdot\|_2$ topology of B_2 for $\mu \in \Omega$. Since Ω is compact there is a positive number $M(y)$ such that

$$\| X_0(\mu) y \| \le M(y), \ \mu \in \Omega . \tag{1.12}$$

The principle of uniform boundedness [3] may then be invoked to show that there is a positive number M such that

$$\| X_0(\mu) \| \le M, \ \mu \in \Omega . \tag{1.13}$$

Using (1.13) together with the fact that the inequalities (1.4) and (1.5) for F and G are required to hold uniformly for $\mu \in \Omega$, $0 \le s \le t \le T$, it is immediately evident upon examination of the method of successive approximations that T can be chosen independently of $\mu \in \Omega$ and that (1.11) holds uniformly for $\mu \in \Omega$, $t \in [0,T]$. Moreover, there is a positive number $K \ge M$ such that

(1.14) $$\|X_n(\mu, t)\| \le K, \quad \mu \in \Omega, \quad t \in [0,T], \quad n = 0, 1, 2, \ldots .$$

For details of the method of successive approximations we suggest [1] .

Let $\mu_0 \in \Omega$, $0 \le s_0 \le t_0$. Since $X_0(\mu, t) \equiv X_0(\mu)$ converges strongly to $X_0(\mu_0)$ as μ converges to μ_0 in Ω we may use (i) to see that for each $y \in B_1$

(1.15) $$\lim_{\substack{\mu \to \mu_0 \\ t \to t_0}} G(X_0(\mu), \mu, t)\, y = G(X_0(\mu_0), \mu_0, t_0)\, y$$

and

(1.16) $$\lim_{\substack{\mu \to \mu_0 \\ t \to t_0 \\ s \to s_0}} F(X_0(\mu, s), \mu, t, s) y = F(X_0(\mu_0, s_0), \mu_0, t_0, s_0)\, y \;. \qquad 0 \le s \le t$$

Combining (1.10) with (1.15) we see that

(1.17) $$\lim_{\substack{\mu \to \mu_0 \\ t \to t_0}} X_1(\mu, t) y = X_1(\mu_0, t_0)\, y, \quad y \in B_1 ,$$

if and only if, for all $y \in B_1$,

$$(1.18) \qquad \lim_{\substack{\mu \to \mu_0 \\ t \to t_0}} \int_0^t F(X_0(\mu, s), \mu, t, s)\, y\, ds = \int_0^{t_0} F(X_0(\mu_0, s), \mu_0, t_0, s)\, y\, ds .$$

Let $t_1 > t_0$ be fixed and let $\hat{F} = F$, $0 \le s \le t \le t_1$, $\hat{F} = 0$, $t < s \le t_1$. Then the continuity expressed by (1.16) implies that for $0 \le s \le t_1$, $s \neq t_0$,

$$(1.19) \qquad \lim_{\substack{t \to t_0 \\ \mu \to \mu_0}} \hat{F}(X_0(\mu, s), \mu, t, s)\, y = \hat{F}(X_0(\mu_0, s), \mu_0, t_0, s)\, y .$$

The continuity (1.16) together with the compactness of the set: $\mu \in \Omega$, $0 \le s \le t \le t_1$, implies the boundedness of $\hat{F}$ in that set. We may then apply the Lebesgue dominated convergence theorem to the integral

$$\int_0^{t_1} \hat{F}(X_0(\mu, s), \mu, t, s)\, y\, ds$$

to obtain the desired result (1.18). The Lebesgue dominated convergence theorem for vector valued functions is proved in [3]. We have noted that (1.17) is implied by (1.18) and hence $X_1(\mu, t)$ is strongly continuous for $\mu \in \Omega$, $0 \le t \le T$.

One now proceeds by induction, showing, just as above, that the strong continuity of $X_n(\mu, t)$ implies that of $X_{n+1}(\mu, t)$ via the equation (1.10). Thus $X_n(\mu, t)$ is strongly continuous for $\mu \in \Omega$, $0 \le t \le T$, $n = 0, 1, 2, \ldots$.

Now let $\varepsilon > 0$ be chosen and let $y \in B_1$. Let N_ε be chosen so that

$$(1.20) \qquad \| X(\mu, t) - X_n(\mu, t) \| \le \frac{\varepsilon}{4 \| y \|_1}, \quad \mu \in \Omega, \quad t \in [0, T],$$

for all $n \ge N_\varepsilon$. Then let μ, t be chosen close enough to μ_0, t_0 in the product topology of $\Omega \otimes [0, T]$ so that

(1.21) $$\|(X_{N_\varepsilon}(\mu_0, t_0) - X_{N_\varepsilon}(\mu, t))y\|_2 \le \varepsilon/2$$

Then we readily verify that

(1.22) $$\|X(\mu_0, t_0)y - X(\mu, t)y\| \le \varepsilon ,$$

and the proof of Theorem 1 is complete.

The result expressed by Theorem 1 is purely local in that strong continuity of $X(\mu, t)$ persists only over a sufficiently short interval $[0, T]$. The example (0.8) offered in the introduction shows that in general we cannot expect more. However, we can obtain global results if we assume a certain boundedness of the solutions.

<u>Theorem 2</u> <u>Let it be known that, for all</u> $\mu \in \Omega$, $X(\mu, t)$ <u>exists and satisfies</u> (1.6) <u>for all</u> $t \ge 0$. <u>A necessary and sufficient condition in order that</u> $X(\mu, t)$ <u>should be strongly continuous in the set:</u> $\mu \in \Omega$, $t \ge 0$ <u>is that there exists a non-negative function</u> $K(t)$ <u>(w.l.o.g. increasing) such that</u>

(1.23) $$\|X(\mu, t)\| \le K(\tau), \ \mu \in \Omega, \ 0 \le t \le \tau < \infty .$$

<u>Proof.</u> The condition is clearly necessary. If, for each $y \in B_1$, $X(\mu, t)y$ is continuous in the compact set: $\mu \in \Omega$, $t \in [0, \tau]$, then $\|X(\mu, t)y\|_2$ is bounded there. The principal of uniform boundedness then shows that $\|X(\mu, t)\|$ is bounded in that set.

Now we will show that the boundedness (1.23) is sufficient for global strong continuity of $X(\mu, t)$. Let $\tau > 0$ be chosen and let $t_0 \in [0, \tau)$. For $t_0 \le t \le \tau$ we have

(1.24) $$X(\mu, t) = \hat{G}(\mu, t_0, t) + \int_{t_0}^{t} F(X(\mu, s), \mu, t, s)\, ds$$

where

(1.25) $$\hat{G}(\mu, t_0, t) = G(X_0(\mu), \mu, t) + \int_0^{t_0} F(X(\mu, s), \mu, t, s)\, ds .$$

Let us assume that $X(\mu, s)$ is strongly continuous in the set: $\mu \in \Omega$, $0 \le s \le t_0$. Now Property (i) implies the boundedness of $G(0,\mu, t)$ and $F(0, \mu, t, s)$, uniformly for $0 \le s \le t \le \tau$. Combining this with the a priori bound (1.23) and Property (ii) we obtain a bound on $G(X_0(\mu), \mu, t)$ and $F(X(\mu, s), \mu, t, s)$ which is uniformly valid for $\mu \in \Omega$, $0 \le s \le t \le \tau$. The Lebesgue dominated convergence theorem implies that $\hat{G}(\mu, t_0, t)$ is strongly continuous for $\mu \in \Omega$, $0 \le t \le \tau$, for the integral in (1.25) involves $X(\mu, s)$ only for $0 \le s \le t_0$. We may now apply the techniques of Theorem 1, altered only very slightly, to extend the strong continuity from $[0, t_0]$ to $[0, t_0 + \hat{T}]$, where the size of $\hat{T} > 0$ depends only upon τ, not upon t_0. Starting with $t_0 = T$, the interval length found in Theorem 1, a finite number of extensions cover $[0,\tau]$ and the proof is complete.

We end this section by noting that the case of a sequence $X_k(t)$ of solutions (as for example (0.8) - (0.10)) is included in Theorems 1 and 2 by taking $\Omega = \{1, 2, \ldots, \infty\}$ with a neighborhood system described by:

(i) N is a neighborhood of the finite integer n if N is any subset of Ω which includes n.

(ii) N is a neighborhood of ∞ if N is a subset of Ω which contains all but finitely many n.

Clearly Ω , as thus topologized, is compact.

2. Application to Quadratic Optimal Control Problems.

Let A be a normal operator defined on a Hilbert space H_1 with spectrum contained in some left half plane:

$$\sigma(A) \subseteq \{\mu \mid \mathrm{Re}(\mu) \leq \mu_0\} . \tag{2.1}$$

Let H_2 be a second Hilbert space and $\tilde{B}: H_2 \to H_1$ a bounded linear operator which, for some $\lambda \notin \sigma(A)$, can be written in the form

$$\tilde{B} = (A - \lambda I)^{-1} B , \tag{2.2}$$

where $B: H_2 \to H_1$ is also bounded. We consider the linear ordinary differential equation

$$\frac{dx}{dt} = Ax + \tilde{B}u \tag{2.3}$$

in H_1 with initial condition

$$x(0) = x_0 \in \Delta = \mathrm{dom}(A) . \tag{2.4}$$

Given a fixed time $T_1 > 0$ let us consider the problem of finding a control $u_*(t)$ lying in the set of measurable vector-valued functions

$$\{u : R^1 \to H_2 \mid \int_0^{T_1} \|u(t)\|^2 \, dt < \infty\} \tag{2.5}$$

which minimizes

$$C(u) = \int_0^{T_1} \{((A - \lambda I)x(t), W(A - \lambda I)x(t)) + (u(t), Uu(t))\} \, dt + ((A - \lambda I)x(T), G(A - \lambda I)x(T)) \tag{2.6}$$

with respect to all controls in (2.5). Here W, U and G are bounded self-adjoint operators. W and G are required to be positive semi-definite while U is to be positive definite. Because (2.2) implies $x(t) \in \Delta$ for all t, $C(u)$ is always defined.

The above problem has been considered in detail by D. L. Lukes and the present author in [4], where it is shown that the minimizing control $u_*(t)$ is generated by the feedback law

(2.7) $$u_*(t) = -U^{-1}\tilde{B}^*(A^* - \lambda I)\, Q(t)(A - \lambda I)\, x_*(t) \equiv D(t)\, x_*(t)$$

which gives rise to responses $x_*(t)$ satisfying

(2.8) $$\frac{dx_*}{dt} = [A + \tilde{B}D(t)]\, x_*(t) .$$

The bounded linear operator $Q(t): H_1 \to H_1$ is given for $t \le T_1$ as the unique strongly continuous solution of the integral equation

(2.9) $$Q(t) = e^{A^*(T_1-t)} G e^{A(T_1-t)} - \int_t^{T_1} e^{A^*(s-t)} [-W + Q(s)(BU^{-1}B^*)Q(s)]\, e^{A(s-t)}\, ds .$$

An optimal control theory formulated in such an abstract setting is of very little value to the engineer unless it represents a limiting case relative to some approximating sequence of finite dimensional problems. In practice the spectrum of A is usually discrete, consisting of a sequence of complex eigenvalues, and the corresponding eigenvectors form a complete orthonormal set in H_1. Thus, a natural way to define finite dimensional approximations is to restrict attention to finite dimensional subspaces of H_1 spanned by finitely many of the eigenvectors of A . H_2 may already be finite dimensional (this is usually the case) but if not it will also be necessary to consider finite dimensional subspaces of it.

We arrange the eigenvalues of A in a sequence, $\lambda_1, \lambda_2, \lambda_3, \ldots,$ and we let E_k denote the orthogonal projection from H_1 onto the space H_{1k} spanned by the eigenvectors of A corresponding to the first k eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. The projections E_k commute with A and converge strongly to the identity as $k \to \infty$. Also, we let $\{\tilde{E}_k\}$ be a similar collection of orthogonal projections on H_2 which commute with the self-adjoint operator U . For $k = 1, 2, 3, \ldots$ we put

(2.10) $$T_k = E_k T E_k,\ E_k T\tilde{E}_k,\ \tilde{E}_k T E_k \text{ or } \tilde{E}_k T\tilde{E}_k$$

according as $T: H_1 \to H_1$, $T: H_2 \to H_1$, $T: H_1 \to H_2$ or $T: H_2 \to H_2$, respectively. Also, we set

(2.11) $$x_k = E_k x, \quad u_k = \tilde{E}_k u \ .$$

We consider now the finite dimensional systems

(2.12) $$\frac{dx_k}{dt} = A_k x_k + \tilde{B}_k u_k , \quad k = 1, 2, 3, \ldots ,$$

with initial conditions

(2.13) $$x_k(0) = x_{0k} \ .$$

One then seeks for a control $u_{k*}(t)$ minimizing

(2.14) $$C_k(u_k) = \int_0^{T_1} \{((A_k - \lambda E_k)x_k(t),\ W_k(A_k - \lambda E_k)x_k(t)) + (u_k(t),\ U_k u_k(t))\} dt + ((A_k - \lambda E_k)x_k(T_1),\ G_k(A_k - \lambda E_k)x_k(T_1)) \ .$$

Again one shows that the minimizing control $u_{k*}(t)$ is generated by

$$u_{k*}(t) = -U_k^{-1}\tilde{B}_k^*(A_k^* - \lambda E_k)\,\hat{Q}_k(t)\,(A_k - \lambda E_k)\,x_{k*}(t)\ , \ = \hat{D}_k(t)\,x_{k*}(t) \tag{2.15}$$

yielding responses $x_{k*}(t)$ satisfying

$$\frac{dx_{k*}}{dt} = [A_k + \tilde{B}_k\,\hat{D}_k(t)]\,x_{k*}(t)\ . \tag{2.16}$$

Here $\hat{Q}_k(t)$ satisfies

$$\hat{Q}_k(t) = e^{A_k^*(T_1 - t)}\,G_k e^{A_k(T_1 - t)} - \int_t^{T_1} e^{A_k^*(s-t)}[-W_k + \hat{Q}_k(s)\,(B_k\,U_k^{-1}\,B_k^*)\,\hat{Q}_k(s)]\,s^{A_k(s-t)}\,ds\ . \tag{2.17}$$

It should be emphasized that in general $\hat{Q}_k \neq Q_k$, $\hat{D}_k \neq D_k$, u_{k*}, and $x_{k*} \neq x_{*k}$.

The real test of this approximation procedure is provided by asking if, for $0 \le t \le T_1$,

$$\lim_{k \to \infty} u_{k*}(t) = u_*(t)\ , \tag{2.18}$$

$$\lim_{k \to \infty} (A_k - \lambda E_k)\,x_{k*}(t) = (A - \lambda I)\,x_*(t)\ . \tag{2.19}$$

Evidently both of these will be true if, again for $0 \le t \le T_1$,

$$\lim_{k \to \infty} \hat{Q}_k(t)\,y = Q(t)y\ , \quad y \in H_1 \tag{2.20}$$

i.e. if $\hat{Q}_k(t)$ converges strongly to $Q(t)$ as $k \to \infty$.

If we reverse the time sense in equations (2.9) and (2.17) and attach an index " ∞ " to $Q(t)$, the equations (2.17), $k = 1, 2, \ldots$ and (2.9) satisfy all of the conditions set down in section 1 . The parameter space

Ω consists of $1, 2, 3, \ldots, \infty$ topologized as indicated at the end of section 1 . Theorem 1 then implies the validity of (2.20) for t sufficiently close to T_1, $t \leq T_1$. The result is obtained in the complete interval $0 \leq t \leq T_1$ by showing that $\|\hat{Q}_k(t)\|$, $\|Q(t)\|$ are uniformly bounded for $0 \leq t \leq T_1$, thus permitting the application of Theorem 2 . This boundedness is rigorously demonstrated in [4] and is a consequence of the observation that the quadratic forms $(y, Q(t)y)$, $(y_k, \hat{Q}_k(t)y_k)$ represent the minimum costs associated with problems of the above type posed on the interval $[t, T_1]$ with initial conditions $x(t) = (A - \lambda I)^{-1} y$, $x_k(t) = (A_k - \lambda E_k)^{-1} y_k$.

References

[1] E. A. Coddington and N. Levinson: "Theory of Ordinary Differential Equations," McGraw-Hill Book Co., New York, 1955.

[2] F. E. Browder: "Nonlinear Equations of Evolution," Ann. of Math. 80 (1964), pp. 485-523.

[3] E. Hille and R. S. Phillips: "Functional Analysis and Semi-groups," American Mathematical Society Colloquium Publications, Vol. 31, 1957.

[4] D. L. Lukes and D. L. Russell: "The Quadratic Criterion for Distributed Systems," Technical Summary Report No. 860, Mathematics Research Center, U. S. Army, University of Wisconsin, Madison, March 1968.