CHARACTERIZATIONS OF REAL MATRICES
OF MONOTONE KIND

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An m by n real matrix $A$ is said to be of monotone kind if

$$Ax \geq 0 \implies x \geq 0.$$  \hspace{1cm} (1)

Collatz [2] treats square matrices of monotone kind and shows that for such matrices the above implication is equivalent to: $A^{-1}$ exists and $A^{-1} \geq 0$. 3) Matrices of monotone kind have useful applications in numerical analysis [2,7].

It is the purpose of this note to generalize Collatz's result to rectangular matrices, and also to show that, for the general rectangular case, a matrix of monotone kind can be further characterized as one for which the convex conical hull of the rows contains the nonnegative orthant.

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3) That is, each element of $A^{-1}$ is nonnegative.
(For an $m$ by $n$ matrix $A$, the convex conical hull of the rows of $A$ is defined as

$$K(A) = \{ z \mid z = A^T u, \ u \geq 0 \}.$$ 

The nonnegative orthant $E^n_+$ is defined by

$$E^n_+ = \{ x \mid x \in E^n, \ x \geq 0 \} ,$$

where $E^n$ is the $n$-dimensional real Euclidean space.)

**Theorem 1.** Let $A$ be an $m$ by $n$ real matrix. Then the following two statements are equivalent:

(2) $A$ has a nonnegative left inverse. In other words, there exists an $n$ by $m$ matrix $Y \geq 0$ such that $YA = I$.

(3) $K(A) \supseteq E^n_+$

**Proof.** Clearly (2) holds if and only if each row $I_i$ of the identity matrix $I$ of order $n$ is a nonnegative linear combination of the rows of $A$.

But this is equivalent to the statement that each unit vector is contained in $K(A)$, which is the case if and only if (3) holds. Q.E.D.

Of course, if $A$ is square, either (2) or (3) is equivalent to $A$ being nonsingular and $Y = A^{-1}$ being nonnegative.

It can be shown by elementary arguments that (1) and (2) are equivalent for a square matrix $A$, and that (2) implies (1) for a general rectangular
(For an $m$ by $n$ matrix $A$, the convex conical hull of the rows of $A$ is defined as

$$K(A) = \{ z \mid z = A^T u, \ u \geq 0 \}.$$ 

The nonnegative orthant $E^+_n$ is defined by

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where $E^n$ is the $n$-dimensional real Euclidean space.)

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Of course, if $A$ is square, either (2) or (3) is equivalent to $A$ being nonsingular and $Y = A^{-1}$ being nonnegative.

It can be shown by elementary arguments that (1) and (2) are equivalent for a square matrix $A$, and that (2) implies (1) for a general rectangular
matrix $A$. The proof that (1) implies (2) for a general rectangular $A$ seems to require the use of either the duality theory of linear programming or a theorem of the alternative for linear inequalities, such as Motzkin's theorem [4, 5, 8]. (Theorems of the alternative may be considered a consequence of the separation theorem for convex sets [1].)

**Theorem 2.** For any $m$ by $n$ real matrix $A$, (1) and (2) are equivalent.

**Proof.** If (2) holds, then $Ax \geq 0$ implies that $x = YAx \geq 0$, and (1) is established.

If (1) holds, then $A$ must be of rank $n$. For, $Ax = 0$ implies that $Ax \geq 0$ and $A(-x) \geq 0$, and hence by (1), $x = 0$, and the rank of $A$ is $n \leq m$.

Thus if (1) holds and $A$ is square ($m = n$), it is nonsingular, and (1) together with $AA^{-1} = I \geq 0$ imply that $A^{-1} \geq 0$.

For $m \geq n$ a different argument is required. We note that $Ax \geq 0, I_1 x < 0$ has no solution for each $i = 1, \ldots, n$. By Motzkin's theorem [4, 5, 8] it follows that $yA = I_1, y \geq 0$ has a solution for each $i$, and (2) follows. Q.E.D.

An alternate proof that (1) implies (2) may be based on the duality theory of linear programming [6] instead of on Motzkin's theorem. If (1) holds then

$$\min_{x} \{I_1 x : Ax \geq 0\} = 0$$

for each $i = 1, \ldots, n$. 

By the duality theory of linear programming [6]

\[
\text{maximum } \{ 0 \mid yA = I_1, \ y \geq 0 \} = 0 \text{ for each } i = 1, \ldots, n,
\]

where the zero denotes an \( m \) vector of zeros. Hence for each \( i = 1, \ldots, n \), \( yA = I_1, \ y \geq 0 \), has a solution. This establishes (2).

Remark. For square matrices, because \((A^{-1})^T = (A^T)^{-1}\), it follows from (2) above that any of the statements (1), (2) or (3) above is equivalent to any of the three statements below:

(1') \( A^T y \geq 0 \implies y \geq 0 \).

(2') \( (A^{-1})^T \) exists and \( (A^T)^{-1} \geq 0 \).

(3') \( K(A^T) \supset E_n^+ \).

Rectangular Matrices of Monotone Kind with Respect to Another Matrix:

Let \( A \) be an \( m \) by \( n \) real matrix and let \( B \) be a \( k \) by \( n \) real matrix. Then the following are equivalent:

(1'') \( Ax \geq 0 \implies Bx \geq 0 \)

(2'') \( YA = B, \ Y \geq 0 \)

(3'') \( K(A) \supset K(B) \)

The equivalence of the above three statements is established by replacing \( I \) by \( B \) or \( B^T \) in the proofs of Theorems 1 and 2 (omitting in the latter case, the demonstration that \( A \) is of full rank and the special argument for nonsingular \( A \)).
Finally it should be remarked that if we define the polar cone of the rows of a matrix $A$ as

$$P(A) = \{x | Ax \geq 0 \},$$

then (1") above can be stated as

$$(1'') \quad P(A) \subseteq P(B).$$

The equivalence of (1") and (3") follows then directly from the duality theorem for polyhedral convex cones of Goldman and Tucker [3, lemma 2].

**Example.** Consider the following $m$ by 2 matrix ($m \geq 2$)

$$A = \begin{bmatrix}
    r_1 \cos \theta_1 & r_1 \sin \theta_1 \\
    \vdots & \vdots \\
    \vdots & \vdots \\
    \vdots & \vdots \\
    r_m \cos \theta_m & r_m \sin \theta_m
\end{bmatrix},$$

where $r_i \geq 0$, $-\pi \leq \theta_i \leq \pi$, for $i = 1, \ldots, m$. Our necessary and sufficient condition (3) (that $A$ be of monotone kind (1) or have a nonnegative left inverse (2)) becomes this: there exist $i, j, i \neq j$, such that for all $k \neq i$, $k \neq j$ ($1 \leq i, j, k \leq m$) we have that

$$r_i > 0, \quad r_j > 0, \quad \theta_j \leq \theta_k \leq \theta_i,$$

$$\frac{\pi}{2} \leq \theta_i - \theta_j < \pi, \quad -\frac{\pi}{2} < \theta_j \leq 0, \quad \frac{\pi}{2} \leq \theta_i < \pi.$$
If $A$ is a 2 by 2 matrix, then $i = 1$ or 2, $j = 1$ or 2, $i \neq j$, and the above condition is necessary and sufficient for $A^{-1}$ to exist and $A^{-1} \geq 0$.

We have then

$$A^{-1} = \frac{1}{\sin (\theta_2 - \theta_1)} \begin{bmatrix}
\frac{\sin \theta_2}{r_1} & -\frac{\sin \theta_1}{r_2} \\
\frac{-\cos \theta_2}{r_1} & \frac{\cos \theta_1}{r_2}
\end{bmatrix} \geq 0.$$
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REFERENCES


