NUMERICAL STUDIES OF VISCOUS, INCOMPRESSIBLE FLOW FOR ARBITRARY REYNOLDS NUMBER

by Donald Greenspan

APPENDIX:
PROGRAMMING VISCOUS, INCOMPRESSIBLE FLOW PROBLEMS

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1. Introduction

The development of the high speed digital computer has resulted in extensive efforts to solve numerically fluid problems whose equations of motion are the Navier-Stokes equations (see, e.g., references [1]-[5] and the additional references contained therein). The interest in these equations is founded not only on the fact that they incorporate boundary layer phenomena, but also on the important observation that they result from both microscopic and macroscopic approaches to viscous flow [6], [7].

In this paper we will adapt a new numerical method [3] to study a prototype problem of the circulation of a steady, viscous, incompressible flow within a square boundary. Both the stream function and its normal derivative will be prescribed on the boundary. The discussion will be self contained and the numerical method will apply equally well to comparable boundary value problems. We will consider with equal ease cases in which the Reynolds number is small ($\mathcal{R} = 10$) and cases in which the
Reynolds number is large \((R = 10^5)\). If and when such steady state flows exist, which is still usually an open matter, the method to be described is vastly more economical and accurate than time dependent, step-by-step methods. The power of our method is contained in the structure of the difference equations which, for all \(R\), yield diagonally dominant systems of linear algebraic equations.

2. **Statement of the Analytical Problem.**

The problem to be considered can be formulated as follows. Let the points \((0, 0)\), \((1, 0)\), \((1, 1)\) and \((0, 1)\) be denoted by \(A\), \(B\), \(C\) and \(D\), respectively (see Figure 2.1). Let \(S\) be the square whose vertices are \(A\), \(B\), \(C\), \(D\) and denote its interior by \(R\). On \(R\) the equations of motion to be satisfied are the two dimensional, steady state, Navier-Stokes equations, that is

\[
\Delta \psi = -\omega \tag{2.1}
\]

\[
\Delta \omega + R \left( \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} \right) = 0 \tag{2.2}
\]

where \(\psi\) is the stream function, \(\omega\) is the vorticity, and \(R\) is the Reynolds number. On \(S\) the boundary conditions to be satisfied are

\[
\psi = 0, \quad \frac{\partial \psi}{\partial x} = 0 \quad \text{on AD} \tag{2.3}
\]

\[
\psi = 0, \quad \frac{\partial \psi}{\partial y} = 0 \quad \text{on AB} \tag{2.4}
\]

\[
\psi = 0, \quad \frac{\partial \psi}{\partial x} = 0 \quad \text{on BC} \tag{2.5}
\]

\[
\psi = 0, \quad \frac{\partial \psi}{\partial y} = -1 \quad \text{on CD} \tag{2.6}
\]
The analytical problem is defined on $R + S$ by (2.1) - (2.6) and is shown diagramatically in Figure 2.1.

Figure 2.1

3. **Difference Approximations.**

Because the numerical method to be developed will be a finite difference method, it will be convenient in this section to recall or to develop several useful finite difference approximations.
First, for $h > 0$, consider the five points $(x, y)$, $(x+h, y)$, $(x, y+h)$, $(x-h, y)$ and $(x, y-h)$, numbered 0, 1, 2, 3 and 4, respectively, in Figure 3.1. For convenience, any function $u(x, y)$ defined at a point numbered 1 will be denoted at that point by $u_1$. Now, if $\omega(x, y)$ is defined at the point numbered 0 in Figure 3.1, then (2.1) can be approximated at 0 by the well known [8] Poisson difference analogue

$$-4\psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 = -h^2 \omega_0.$$
On the other hand, if \( \psi(x, y) \) is defined at the points numbered 0, 1, 2, 3, 4 in Figure 3.1, then (2.2) can be approximated [3] as follows. Set

\[
\alpha = \psi_1 - \psi_3 \\
\beta = \psi_2 - \psi_4
\]

and at the point numbered 0 in Figure 3.1 approximate (2.2) by

\[
(3.2a) \quad (-4 - \frac{\alpha \rho}{2} - \frac{\beta \rho}{2}) \omega_0 + \omega_1 + (1 + \frac{\alpha \rho}{2}) \omega_2 + (1 + \frac{\beta \rho}{2}) \omega_3 + \omega_4 = 0, \\
\text{if } \alpha \geq 0, \beta \geq 0;
\]

\[
(3.2b) \quad (-4 - \frac{\alpha \rho}{2} + \frac{\beta \rho}{2}) \omega_0 + \omega_1 + (1 - \frac{\beta \rho}{2}) \omega_2 + (1 + \frac{\alpha \rho}{2}) \omega_3 + \omega_4 = 0, \\
\text{if } \alpha \geq 0, \beta < 0;
\]

\[
(3.2c) \quad (-4 + \frac{\alpha \rho}{2} - \frac{\beta \rho}{2}) \omega_0 + \omega_1 + \omega_2 + (1 + \frac{\beta \rho}{2}) \omega_3 + (1 - \frac{\alpha \rho}{2}) \omega_4 = 0, \\
\text{if } \alpha < 0, \beta \geq 0;
\]

\[
(3.2d) \quad (-4 + \frac{\alpha \rho}{2} + \frac{\beta \rho}{2}) \omega_0 + \omega_1 + \omega_2 + \omega_3 + (1 - \frac{\alpha \rho}{2}) \omega_4 = 0, \\
\text{if } \alpha < 0, \beta < 0.
\]

Next, recall that for three points \((x, y), (x+h, y), (x+2h, y)\), numbered 0, 1, 2, respectively, in Figure 3.2(a), one has the approximation [9]

\[
(3.3a) \quad \frac{\partial \psi}{\partial x} \bigg|_0 = \frac{1}{2h} (-3\psi_0 + 4\psi_1 - \psi_2);
\]

for three points \((x, y), (x, y+h), (x, y+2h)\), numbered 0, 1, 2, respectively, in Figure 3.2(b), one has the approximation

\[
(3.3b) \quad \frac{\partial \psi}{\partial y} \bigg|_0 = \frac{1}{2h} (-3\psi_0 + 4\psi_1 - \psi_2);
\]
for three points \((x, y), (x-h, y), (x-2h, y)\), numbered \(0, 1, 2\), respectively, in Figure 3.2(c), one has the approximation

\[
\frac{\partial \psi}{\partial x} \bigg|_0 = \frac{1}{2h} \left(3\psi_0 - 4\psi_1 + \psi_2\right);
\]

and that for three points \((x, y), (x, y-h), (x, y-2h)\), numbered \(0, 1, 2\), respectively, in Figure 3.2(d), one has

\[
\frac{\partial \psi}{\partial y} \bigg|_0 = \frac{1}{2h} \left(3\psi_0 - 4\psi_1 + \psi_2\right).
\]

Finally, let us develop approximations for the Laplace operator

\[
\psi_{xx} + \psi_{yy}
\]

on \(S\) in terms of certain function values and normal derivatives.

\[\text{Figure 3.2}\]

\[\text{Figure 3.3}\]
Consider the four points \((x, y), (x+h, y), (x, y+h), (x, y-h)\), numbered 0, 1, 2, 4, respectively, in Figure 3.3(a), and let us try to determine parameters 
\(\alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_5\) such that

\[ (\psi_{xx} + \psi_{yy}) \bigg|_0 = \alpha_0 \psi_0 + \alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_4 \psi_4 + \alpha_5 \left( \frac{\partial \psi}{\partial x} \right) \bigg|_0. \]

In \((3.4)\), expansion of \(\psi_1, \psi_2\) and \(\psi_4\) into Taylor series about the point numbered 0 and reorganization of terms implies

\[ (\psi_{xx} + \psi_{yy}) \bigg|_0 = \psi_0 \left( \alpha_0 + \alpha_1 + \alpha_2 + \alpha_4 \right) \]

\[ + \psi_x \left( h\alpha_1 + \alpha_5 \right) \]

\[ + \psi_y \left( h\alpha_2 - h\alpha_4 \right) \]

\[ + \psi_{xx} \left( \frac{h^2}{2} \alpha_1 \right) \]

\[ + \psi_{yy} \left( \frac{h^2}{2} \alpha_2 + \frac{h^2}{2} \alpha_4 \right) \]

\[ + \ldots \ldots \ldots \ldots \ldots \ldots \]

In this latter equality, the setting of corresponding terms equal yields

\[ \alpha_0 + \alpha_1 + \alpha_2 + \alpha_4 = 0 \]

\[ h\alpha_1 + \alpha_5 = 0 \]

\[ h\alpha_2 - h\alpha_4 = 0 \]

\[ \frac{h^2}{2} \alpha_1 = 1 \]

\[ \frac{h^2}{2} \alpha_2 + \frac{h^2}{2} \alpha_4 = 1, \]
the solution of which is

\[ \alpha_0 = \frac{4}{h^2}, \alpha_1 = \frac{2}{h^2}, \alpha_2 = \alpha_4 = \frac{1}{h^2}, \alpha_5 = -\frac{2}{h}. \]

Thus one arrives at the following approximation:

\[ (3.5a) \quad (\psi_{xx} + \psi_{yy}) |_0 = -\frac{4}{h^2} \psi_0 + \frac{2}{h^2} \psi_1 + \frac{1}{h^2} \psi_2 + \frac{1}{h^2} \psi_4 - \frac{2}{h} \left( \frac{\partial \psi}{\partial x} \right) |_0. \]

Similarly, for the four points \((x, y), (x+h, y), (x, y+h), (x-h, y),\) numbered 0, 1, 2, 3, respectively, in Figure 3.3(b), one has

\[ (3.5b) \quad (\psi_{xx} + \psi_{yy}) |_0 = -\frac{4}{h^2} \psi_0 + \frac{1}{h^2} \psi_1 + \frac{2}{h^2} \psi_2 + \frac{1}{h^2} \psi_3 - \frac{2}{h} \left( \frac{\partial \psi}{\partial y} \right) |_0; \]

for the four points \((x, y), (x, y+h), (x-h, y), (x, y-h),\) numbered 0, 2, 3, 4, respectively, in Figure 3.3(c), one has

\[ (3.5c) \quad (\psi_{xx} + \psi_{yy}) |_0 = -\frac{4}{h^2} \psi_0 + \frac{1}{h^2} \psi_1 + \frac{2}{h^2} \psi_2 + \frac{1}{h^2} \psi_3 + \frac{2}{h} \left( \frac{\partial \psi}{\partial x} \right) |_0; \]

and for the four points \((x, y), (x+h, y), (x-h, y), (x, y-h),\) numbered 0, 1, 3, 4, respectively, in Figure 3.3(d), one has

\[ (3.5d) \quad (\psi_{xx} + \psi_{yy}) |_0 = -\frac{4}{h^2} \psi_0 + \frac{1}{h^2} \psi_1 + \frac{1}{h^2} \psi_3 + \frac{2}{h^2} \psi_4 + \frac{2}{h} \left( \frac{\partial \psi}{\partial y} \right) |_0. \]

Note that the numbering of the points in Figure 3.3 is consistent with that in Figure 3.1.


For a fixed positive integer \(n\), set \(h = \frac{1}{n}\). Starting at \((0, 0)\) with grid size \(h\), construct and number in the usual way [8] the set of interior grid points \(R_h\) and the set of boundary grid points \(S_h\). To within some
preassigned tolerance $\epsilon$, we aim to find a solution $\psi^{(k)}$ of (3.1) on $R_h$ and a solution $\omega^{(k)}$ of (3.2a) - (3.2d) on $R_h + S_h$, subject to the boundary restrictions on $\psi$, and we proceed as follows.

Denote by $R_{h,1}$ those points of $R_h$ whose distance from $S$ is $h$, and denote by $R_{h,2}$ those points of $R_h$ whose distance from $S$ is greater than $h$. Initially, set

\begin{align}
\psi^{(0)} &= C_1, \quad \text{on } R_h \\
\omega^{(0)} &= C_2, \quad \text{on } R_h + S_h,
\end{align}

where $C_1$ and $C_2$ are constants. A modified over-relaxation procedure which does not require much storage to obtain the desired result is then applied as follows to yield $\psi^{(1)}$ from $\psi^{(0)}$ and $\omega^{(0)}$. On $R_h$, set

\begin{equation}
\psi^{(1,0)} = \psi^{(0)}
\end{equation}

and on $R_{h,2}$ generate $\psi^{(1,1)}$ by sweeping along each row of $R_{h,2}$ from left to right, starting from the bottom row and proceeding to the top row, by the recursion formula

\begin{equation}
\psi_0^{(1,j)} = (1-r_\psi) \psi_0^{(1,j-1)} + \frac{r_\psi}{4} \left[ \psi_1^{(1,j-1)} + \psi_2^{(1,j-1)} + \psi_3^{(1,j)} + \psi_4^{(1,j)} \right],
\end{equation}

where $0 < r_\psi < 1$. After each such sweep, $\psi^{(1,j)}$ is defined on $R_{h,2}$ by the weighted average

\begin{equation}
\psi^{(1,j)} = \xi \psi^{(1,j-1)} + (1-\xi) \psi^{(1,j)}, \quad 0 \leq \xi \leq 1.
\end{equation}
This inner iteration process continues until, for the given tolerance $\epsilon$, one has

\[(4.6) \quad \| \psi^{(1, k)} - \psi^{(1, k+1)} \| < \epsilon, \]

from which one defines on $\mathbb{R}_{h,1}$

\[(4.7) \quad \psi^{(1)} = \psi^{(1, k)}. \]

In order to define $\psi^{(1)}$ on $\mathbb{R}_{h,1}$, we apply (3.3a) - (3.3d) and (2.3) - (2.6) in the following fashion. At each point of $\mathbb{R}_{h,1}$ of the form $(i \, h, h), \ i = 1, 2, \ldots, n-1$, set (in the notation of Figure 3.2b)

\[(4.8a) \quad \psi^{(1)}_1 = \frac{\psi^{(1)}_2}{4}. \]

Similarly, at each point of $\mathbb{R}_{h,1}$ of the form $(h, i \, h), \ i = 2, 3, \ldots, n-2$, set (in the notation of Figure 3.2a)

\[(4.8b) \quad \psi^{(1)}_1 = \frac{\psi^{(1)}_2}{4}, \]

while at each point of $\mathbb{R}_{h,1}$ of the form $(1-h, i \, h), \ i = 2, 3, \ldots, n-2$, set (in the notation of Figure 3.2c)

\[(4.8c) \quad \psi^{(1)}_1 = \frac{\psi^{(1)}_2}{4}. \]

Finally, at each point of $\mathbb{R}_{h,1}$ of the form $(i \, h, 1-h), \ i = 1, 2, \ldots, n-1$, set (in the notation of Figure 3.2d)

\[(4.8d) \quad \psi^{(1)}_1 = \frac{h}{2} + \frac{\psi^{(1)}_2}{4}. \]
Thus, (4.3) and (4.4a) - (4.4d) define $\psi^{(1)}$ on all of $R_h$.

Next, proceed to construct $\omega^{(1)}$ on $R_h + S_h$ as follows. On $S_h$, use (2.1), (2.3)-(2.6) and (3.5a) - (3.5d) to yield at each point $(i h, 0)$, $i = 0, 1, 2, \ldots, n$ (in the notation of Figure 3.3(b))

(4.9a) \[ \bar{\omega}^{(1)}_0 = -\frac{2\psi^{(1)}_2}{h^2} \]

at each point $(0, ih)$, $i = 1, 2, \ldots, n-1$, in the notation of Figure 3.3(a)

(4.9b) \[ \bar{\omega}^{(1)}_0 = \frac{2\psi^{(1)}_1}{h^2} ; \]

at each point $(1, ih)$, $i = 1, 2, \ldots, n-1$, in the notation of Figure 3.3(c)

(4.9c) \[ \bar{\omega}^{(1)}_0 = \frac{2\psi^{(1)}_3}{h^2} \]

and, at each point $(ih, 1)$, $i = 0, 1, 2, \ldots, n$, in the notation of Figure 3.3(d)

(4.9d) \[ \bar{\omega}^{(1)}_0 = \frac{2\psi^{(1)}_4}{h^2} . \]

One then defines $\omega^{(1)}$ on $S_h$ by the weighted average formula

(4.10) \[ \omega^{(1)} = \delta \omega^{(0)} + (1 - \delta) \bar{\omega}^{(1)} , \; 0 \leq \delta \leq 1 . \]

We proceed next to determine $\omega^{(1)}$ on $R_h$ by again using a modified over-relaxation procedure. At each point of $S_h$ set

$\omega^{(1), 0} = \omega^{(1)}$
while at each point of $R_h$ set
\[ \omega(1, 0) = \omega(0) \, . \]

Then generate $\underline{w}(1, 1)$ by sweeping along each row of $R_h$ from left to right, starting from the bottom row and proceeding to the top row, by the recursion formula

\[ (4.11) \quad \underline{w}(l, j) = (1 - r_{\omega}) \omega(l, j-1) + \frac{r_{\omega}}{\Omega_0} \left[ \Omega_1 \cdot \omega(l, j-1) + \Omega_2 \cdot \omega(1, j-1) \right. \]
\[ + \Omega_3 \cdot \underline{w}(l, j) + \Omega_4 \cdot \underline{w}(1, j) \left. \right] \, , \]

where $0 < r_{\omega} < 2$, where

\[
\Omega_0 = 4 + \frac{\alpha}{2} |\alpha| + \frac{\beta}{2} |\beta| \\
\Omega_1 = \begin{cases} 1 & , \beta \geq 0 \\
1 + \frac{\alpha}{2} |\beta| & , \beta < 0 \end{cases} \\
\Omega_2 = \begin{cases} 1 + \frac{\alpha}{2} \alpha & , \alpha \geq 0 \\
1 & , \alpha < 0 \end{cases} \\
\Omega_3 = \begin{cases} 1 + \frac{\beta}{2} \beta & , \beta \geq 0 \\
1 & , \beta < 0 \end{cases} \\
\Omega_4 = \begin{cases} 1 & , \alpha \geq 0 \\
1 + \frac{\beta}{2} |\alpha| & , \alpha < 0 \end{cases}
\]

and where, as defined previously,
\[ \alpha = \psi_1 - \psi_3 \]
\[ \beta = \psi_2 - \psi_4 \]

After each such sweep, \( \omega^{(1, j)} \) is defined on \( R_h \) by the weighted average
\[ \omega^{(1, j)} = \delta \omega^{(1, j-1)} + (1 - \delta) \omega^{(1, j)}, \quad 0 \leq \delta \leq 1 \]

where \( \delta \) is the same weight as that used in (4.10). This inner iteration continues until, for the given tolerance \( \epsilon \), one has
\[ |\omega^{(1, K)} - \omega^{(1, K+1)}| < \epsilon \]

from which one defines on \( R_h \)
\[ \omega^{(1)} = \omega^{(1, K)} \]

Proceed next to determine \( \psi^{(2)} \) on \( R_h \) from \( \omega^{(1)} \) and \( \psi^{(1)} \) in the same fashion as \( \psi^{(1)} \) was determined from \( \omega^{(0)} \) and \( \psi^{(0)} \). Then construct \( \omega^{(2)} \) on \( R_h + S_h \) from \( \omega^{(1)} \) and \( \psi^{(2)} \) in the same fashion as \( \omega^{(1)} \) was determined from \( \omega^{(0)} \) and \( \psi^{(1)} \). In the indicated fashion, construct the finite sequences of outer iterates
\[ \psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(m)} \]
\[ \omega^{(0)}, \omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(m)} \]

which satisfy
\[ |\psi^{(m)} - \psi^{(m+1)}| < \epsilon, \quad \text{on} \ R_h \]
\[ |\omega^{(m)} - \omega^{(m+1)}| < \epsilon, \quad \text{on} \ R_h + S_h \]
The discrete functions $\psi^{(m)}$ and $\omega^{(m)}$ are taken to be the numerical approximations of $\psi(x, y)$ and $\omega(x, y)$, respectively, after verifying that they satisfy (3.1) and (3.2a)-(3.2d).

5. **Examples.**

We will attempt now to summarize the results of the large number of examples run on the CDC 3600 at the University of Wisconsin.

In Figures 5.1 - 5.12 are shown graphically the streamlines and equivorticity curves for $\mathcal{R} = 10, 100, 500, 1000, 3000, 100000$ for the set of parameter values $h = \frac{1}{20}$, $C_1 = C_2 = 0$, $r_\psi = 1.8$, $r_\omega = 1$, $\xi = 0.1$, $\delta = 0.7$. A tolerance of $10^{-4}$ was taken for convergence of both inner and outer iterations. The outer iterations for each of $\mathcal{R} = 10, 500, 1000, 3000, 100000$ converged in fewer than ten minutes and the number of outer iterations required were, respectively, 10, 25, 20, 16, 14. The case $\mathcal{R} = 100$ was allowed only twelve minutes of running time at the end of which 40 iterations had elapsed and convergence to $6 \cdot 10^{-4}$ had resulted.

It was clear that for convergence $\xi$ and $\delta$ depended on $h$. For fixed $C_1 = C_2 = 0$ and $r_\psi = 1.8$, $r_\omega = 1$, the following results were found. Outer iteration convergence was achieved for $h = \frac{1}{8}$, $\xi = \delta = 0$, but outer iteration divergence resulted in every case for $h \leq \frac{1}{10}$, $\xi = \delta = 0$. For
h = \frac{1}{16}, outer iteration convergence was achieved with the choice $\xi = \delta = 0.1$, but outer iteration convergence was greatly accelerated as $\delta$ was allowed to increase. For $h = \frac{1}{20}$, all choices of $\xi \leq 0.1, \delta \leq 0.5$ resulted in outer iteration divergence. Further experimentation into the relationships between $\xi, \delta, h$, convergence, and divergence was deemed to be of great interest but too costly to be run at the present time.

Occasionally, the method did not converge because an inner iteration did not converge. When this happened, invariably the choice of $r_{\omega}$ was at fault and a new choice was made after several trial values were tested. The choices $r_{\psi} = 1.8$ and $r_{\omega} = 1$ were finally decided upon because they worked well uniformly, even though inner iteration convergence could often be accelerated by different choices.

In cases where the outer iterations were diverging, no choices of $C_1$ and $C_2$ ever resulted in convergence.

With regard to the physics of the problem, it should be observed that Figures 5.7 - 5.12 indicate clearly that the vorticity is becoming uniform in a large connected subregion of the given region, as was predicted theoretically by Batchelor [10].
Finally, it should be noted that we are documenting our computations by the inclusion of the computer program in an appendix. This is absolutely necessary if other workers in the field are to be able to duplicate our computations in order to verify or to refute our results. Such an omission in the paper of Burggraf [1] caused us great consternation since our duplication of his work for $R = 0$ yielded divergence while he claimed convergence. In this connection, the recent report of Smith [11] proves theoretically that Burggraf's method must diverge for all sufficiently small $h$. 
FIGURE 5.1 Streamlines for Reynolds number 10.
FIGURE 5.2 Streamlines for Reynolds number 100
FIGURE 5.3 Streamlines for Reynolds number 500
FIGURE 5.4 Streamlines for Reynolds number 1000
FIGURE 5.5 Streamlines for Reynolds number 3000
FIGURE 5.6 Streamlines for Reynolds number 100000
FIGURE 5.7 Equivorticity curves for Reynolds number 10.
FIGURE 5.8 Equivorticity curves for Reynolds number 100
FIGURE 5.9 Equivorticity curves for Reynolds number 500
FIGURE 5.10 Equivorticity curves for Reynolds number 1000
FIGURE 5.11 Equivorticity curves for Reynolds number 3000
FIGURE 5.12 Equivorticity curves for Reynolds number 100000
APPENDIX

PROGRAMMING VISCOUS, INCOMPRESSIBLE FLOW PROBLEMS

by M. McClellan

Definitions of Main Program Variables and Parameters

PSI = stream function vector
W = vorticity vector
XMAX, XMIN = extreme values of x for rectangular region
YMAX, YMIN = " " y " "
R = Reynold's number
H = grid size
OMEGAP = relaxation factor for PSI inner-iterations
OMEGAW = " " W " "
TOL = tolerance for both inner- and outer-iterations
XI = weighting factor for PSI
DELTA = " " W
M = number of vertical lines in the grid
N = " horizontal " "
ITERMAX = maximum number of outer-iterations
ITERMAXP = " " inner-iterations for PSI.
ITERMAXW = " " W
TOLTEST = number of outer-iterations between tests for problem convergence
TOLTESTP = number of PSI inner-iterations between tests for convergence
TOLTESTW = number of W inner-iterations between tests for convergence
BPO = initial value of PSI in interior
BWO = " W "
BP1, BP2, BP3, BP4 = initial values of PSI on right, top, left and bottom boundary lines, resp.
BW1, BW2, BW3, BW4 = initial values of W on right, top, left and bottom boundary lines, resp.
PROGRAM NS4
COMMON/1/M,N,NP,MP,XMIN,XMAX,YMIN,YMAX,H
COMMON/INIT/RP0,RP1,RP2,RP3,RP4,RW0,RW1,RW2,RW3,RW4,M1,N1
DIMENSION PSI(41,41),PSISAV(41,41),W(41,41),WSAV(41,41),AUX(41,41)
TYPE INTEGER TOLTEST,TOLTESTP,TOLTESTWN

C
C READ IN NUMBER OF PROBLEMS.
READ 903, NPROBS
C
C READ IN PROBLEM PARAMETERS.
READ 905, YMIN,XMAX,YMIN,YP1,YP2,YP3,YP4,RP0,RP1,RP2,RP3,RP4,RW0,RW1,RW2,RW3,RW4,M1,N1
C
C PRINT PROBLEM DESCRIPTION AND PARAMETERS.
PRINT 909, XMIN,XMAX,YMIN,YMAX,M,H,OMEGAP,OMEGAW,TOL,TOLTEST,ITERMAX,ITERMAXP,ITERMAXW,1

C
C PRINT 1FLOW IN A CAVITY (UNIT SQUARE) /
C 2 20X,12RANGE OF X =,F6.2,3H TO,F6.2 /
C 3 20X,12RANGE OF Y =,F6.2,3H TO,F6.2 /
C 4 20X,9THE THERE ARE,15,15H HORIZONTAL AND,15,56H VERTICAL LINES COMPR
C 5ISING THE GRID, INCLUDING BOUNDARY. /=20X,15HSTEP SIZE (H) =,F10.5 /
C 6 20X,39RELAXATION FACTOR FOR STREAM FUNCTION =,F6.2 /
C 7 20X,33RELAXATION FACTOR FOR VORTICITY =,F6.2 /
C 8 20X,11HTOLERANCE =,E10.1 / 19X,23H TOLERANCE TEST CYCLE =,I5 /
C 9 20X,30MAXIMUM NUMBER OF ITERATIONS =,I6 /
C 10 20X,17REYNOLDS NUMBER =,F10.2 /
C
C PRINT 901, ITERMAXP,TOLTESTP,ITERMAXW,TOLTESTWN,XI,DELTA

C
C COMPUTE ADDITIONAL PROBLEM PARAMETERS.
M1=M-1
M2=M-2
RMAX=1.0+5
H2=H+H
CP1=1.0-OMEGAP
CP2=0.25*OMEGAP
CP5=0.5+H
C#0=1.0-OMEGAW
C#5=2.0/H2
C#7=2.0/H
C INITIALIZE VECTORS W AND PSI.
   CALL INIT(W,PSI)
C PRINT INITIAL VECTORS W AND PSI.
   PRINT 911, ITER, RMAX
911 FORMAT(/'10X,16HAT ITERATION NO.,16,D20H MAXIMUM RESIDUAL =,E12.4
   1 /20X,15HSTREAM FUNCTION
   CALL PRTMAT(PSI)
   PRINT 912
912 FORMAT(/'20X,9HVORTICITY )
   CALL PRTMAT(W)
C BEGIN MAIN LOOP FOR OUTER ITERATIONS.
C
10  ITER=ITER+1
   ITOL=ITOL+1
   IF (.ITOL, LT, ITOLST) 105,102
C SAVE STREAM FUNCTION FROM PREVIOUS OUTER ITERATION.
102  DO 1021 J=2,N1
      DO 1021 I=2,N1
1021  AUX(I,J)=PSI(I,J)
105  RMAXP=1,E94
   ITERP=0
106  ITOLP=0
12  ITERP=ITERP+1
   ITOLP=ITOLP+1
C COMPUTE ONE SWEEP OF INNER REGION FOR STREAM FUNCTION.
   DO 20 J=3,N2
   DO 20 I=3,N2
20   PSI(I,J)=CP1*PSI(I,J)+CP2*(PSI(I+1,J)+PSI(I,J+1)+PSI(I-1,J)
   1 + PSI(I,J-1)+2*H*(I,J))
   IF (.ITOLP, LT, ITOLSTP) 24,25
C RECALCULATE STREAM FUNCTION IN INNER REGION USING WEIGHTING.
24  DO 241 J=3,N2
   DO 241 I=3,N2
241  PSI(I,J)=X1*PSISAV(I,J)+X1*PSI(I,J)
C BOTTOM INNER BOUNDARY LINE FOR STREAM FUNCTION.
   DO 21 I=2,M1
21   PSI(I,1)=.25*PSI(I,3)
C LEFT AND RIGHT INNER BOUNDARY LINES FOR STREAM FUNCTION.
   DO 22 J=3,M2
   PSI(2,J)=.25*PSI(3,J)
22  PSI(M1,J)=.25*PSI(M2,J)
C TOP INNER BOUNDARY LINE FOR STREAM FUNCTION.
   DO 23 I=2,M1
23   PSI(I,M1)=.25*PSI(I,2)+CS
   GO TO 12
25  RMAX1P=0.0
C
C RECALCULATE STREAM FUNCTION, USING WEIGHTING, AND RESIDUALS
C AN INNER REGION.
DO 207 J=3,N2
DO 207 I=3,M2
PSI(I,J)=XI*PSISAV(I,J)+XI1*PSI(I,J)
RES=ABS(PSI(I,J)-PSISAV(I,J))
IF(RES.GT.RMAX1P) 206,207

C BOTTOM INNER BOUNDARY LINE FOR STREAM FUNCTION AND RESIDUALS.
DO 261 I=2,M1
PSIOLD=PSI(I,2)
PSINEW=.25*PSI(I,3)
PSI(I,2)=PSINEW
RES=ABS(PSINEW-PSIOLD)
IF(RES.GT.RMAX1P) 260,261

C LEFT AND RIGHT INNER BOUNDARY LINES FOR STREAM FUNCTION AND RESIDUALS.
DO 281 J=3,N2
PSIOLD=PSI(2,J)
PSINEW=.25*PSI(3,J)
PSI(2,J)=PSINEW
RES=ABS(PSINEW-PSIOLD)
IF(RES.GT.RMAX1P) 270,271

C TOP INNER BOUNDARY LINE FOR STREAM FUNCTION AND RESIDUALS.
DO 291 I=2,M1
PSIOLD=PSI(I,N1)
PSINEW=.25*PSI(I,N2)+CP5
PSI(I,N1)=PSINEW
RES=ABS(PSINEW-PSIOLD)
IF(RES.GT.RMAX1P) 290,291

C TEST MAXIMUM RESIDUAL OF STREAM FUNCTION FOR DIVERGENCE.
IF(RMAXP.GT.1.E-5) 32,35

C PRINT 9017, RMAXP, ITERP
9017 FORMAT(// 76H **** DIVERGENCE IN PSI ONLY ITERATIONS. PROBLEM AB
1ANNOYED. MAX RESIDUAL =E12,4,8E AT ITER=16 )
9009 FORMAT(// 20X,20HSTREAM FUNCTION, PSI )
          CALL PRTHAT(PSI)
          PRINT 9050
          CALL PRTHAT(W)
          GO TO 70
C TEST MAXIMUM RESIDUAL OF STREAM FUNCTION FOR CONVERGENCE.
C IF STREAM FUNCTION HAS CONVERGED FOR INNER ITERATIONS,
C TEST FOR OUTER -ITERATION CONVERGENCE.
35 IF(RMAXP.LT. TOL) 40,45
40 PRINT 915, ITERP,TOL,RMAXP,ITER
915 FORMAT(20H **** AT ITERATION,16,10H TOLERANCE,E10.1,
   1 34H SATISFIED WITH MAXIMUM RESIDUAL =,E15.6,10H FOR PSI(15,1))
   IF(ITOR.LT. TOLTEST) 50,405
405 PRINT 9009
   CALL PRMTMAT(PSI)
   IF(ITER .GE. 1) GO TO 425
   RMAX1=0.0
C GET MAXIMUM OUTER -ITERATION RESIDUAL FOR STREAM FUNCTION.
   DO 42J=2,N1
   DO 42I=2,M1
   RES=ABS(P(PSI(I,J))-AUX(I,J))
   IF(RES .GT. RMAX1) 41,42
41   RMAX1=RES
42 CONTINUE
   RMAX=RMAX1
425 ITOL=0
   PRINT 9042, ITER,RMAX
9042 FORMAT(20H **** AT ITERATION,16,20H MAXIMUM RESIDUAL =,E15.6,
   1 19H FOR LARGE PROBLEM.)
C TEST OUTER-ITERATION RESIDUAL FOR CONVERGENCE.
   IF(RMAX .LT.1.E-5 ) 432,435
432 PRINT 917, RMAX,ITER
917 FORMAT(20H AT ITER,16 )
   N P=N P=1
   PRINT 9009
   CALL PRMTMAT(PSI)
   PRINT 9050
   CALL PRMTMAT(W)
   GO TO 70
C TEST OUTER-ITERATION RESIDUAL FOR CONVERGENCE.
435 IF(RMAX .LE. TOL) 440,445
440 PRINT 9440, ITER,TOL,RMAX
9440 FORMAT(20H **** AT ITERATION,16,10H TOLERANCE,E10.1,
   1 34H SATISFIED WITH MAXIMUM RESIDUAL =,E15.6 /)
   N P=N P=1
   PRINT 9009
   CALL PRMTMAT(PSI)
   PRINT 9050
   CALL PRMTMAT(W)
   GO TO 70
C TEST IF MAXIMUM NUMBER OF OUTER ITERATIONS EXCEEDED.
445 IF(ITER .GE. ITERMAX) 947,50
947 PRINT 913, RMAX,ITER
913 FORMAT(20H AT ITER,16 )
   N P=N P=1
   PRINT 9009
   CALL PRMTMAT(PSI)
   PRINT 9050
   CALL PRMTMAT(W)
GO TO 70

C TEST IF MAXIMUM NUMBER OF INNER ITERATIONS EXCEEDED FOR STREAM FN.
C IF (ITERP.GE. ITERMXP) 47,106
C PRINT 9013, RMAXP, ITERP

9013 FORMAT(/ 55H **** MAXIMUM NUMBER OF ITERATIONS USED FOR PSI-ONL
        1Y ITERATIONS. MAX RESIDUAL = .E12.4,8H AT ITER,16 )
MXP=NP=1
PRINT 9009
CALL PRMAT(P1)
PRINT 9008
CALL PRMAT(W)
GO TO 70

C BEGIN INNER-ITERATIONS FOR VORTICITY.

50 RMAXW=1.E91
C COMPUTE VORTICITY ON BOUNDARY USING WEIGHTING.
C TOP AND BOTTOM BOUNDARIES.
DO 5072 I=1,M
5072 W(I,N)=DELTA*W(I,N)-DELTA*CH5*(PSI(I,N1)-H)
C LEFT AND RIGHT BOUNDARIES.
DO 5074 J=2,N1
5074 W(I,J)=DELTA*W(I,J)-DELTA*CH5*PSI(2,J)
C COMPUTE RELAXATION COEFFICIENTS FOR VORTICITY IN INTERIOR.
C COMPUTE ONE SWEEP OF VORTICITY IN INTERIOR.

506 ITERW=ITERP+1
ITOLW=ITOLP+1
DO 62 J=2,N1
DO 62 I=2,M1
WSAV(I,J)=W(I,J)
A=PSI(I+1,J)-PSI(I-1,J)
R=PSI(I,J+1)-PSI(I,J-1)
R2A=R2A*A
R2R=R2R*R
IF(A.GE.0.)51,55
IF(B.GE.0.)52,53
52 CW0=4.0+R2A*R2B
CW1=CW4=1.0
CW2=1.0+R2A
CW3=1.0+R2R
GO TO 60
53 CW0=4.0+R2A-R2R
CW3=CW4=1.0
CW2=1.0-R2A
CW1=1.0-R2R
GO TO 60
55 IF(B.GE.0.)56,57
56 CW0=4.0-R2A+R2B
CW1=CW2=1.0
CW3=1.0+R2B
CW4=1.0-R2A

GO TO 60
GO TO 60
57 CM0=4.0-4H2A-4R2B
CM2=CM3=1.0
CM1=1.0-4R2B
CM4=1.0-4R2A
C
60 CM0=OMEGAM/CM0
62 W(I,J)=CM1*W(I,J)+CM2*W(I,J)+CM3*W(I-1,J)
1+CM4*W(I,J-1)
IF(ITOL.LE.1, ITOLTEST) 625, 63
C RECOMPUTE VORTICITY IN INTERIOR USING WEIGHTING.
625 DO 628 J=2,M1
DO 628 I=2,M1
628 W(I,J)=DELTA*KSAV(I,J)+DELTA#W(I,J)
GO TO 506
C
C RECOMPUTE VORTICITY USING WEIGHTING AND RESIDUALS IN INTERIOR.
63 RMAX1=0.0
DO 65 J=2, M1
DO 65 I=2, M1
65 W(I,J)=DELTA#W(I,J)+DELTA#SAV(I,J)+DELTA#W(I,J)
RES=ABS(W(I,J)-SAV(I,J))
IF(RES.GT.RMAX1) 64, 65
64 RMAX1=RES
65 CONTINUE
RMAX=W/RMAX1
C
C TEST VORTICITY FOR INNER-ITERATION CONVERGENCE.
66 IF(RMAX1.GT.1.0*5) 665, 666
665 PRINT 9665, RMAX6, ITER6
9665 FORMAT(//, 5H ****** DIVERGENCE IN W-ONLY ITERATIONS, MAX RESIDUAL
1 =, E12.4, 8H AT ITER, 16 )
MPN=1
PRINT 9001
CALL PRIMAT(PSI)
PRINT 9040
CALL PRIMAT(W)
GO TO 70
C
C TEST VORTICITY FOR INNER-ITERATION CONVERGENCE.
666 IF(RMAX1.LE.TOL1) 67, 675
67 PRINT 9054, ITER6, TOL1, RMAX6, ITER6
9054 FORMAT(//, 6H 20H ****** AT ITERATION, I6, 10H TOLERANCE, E10.1,
1 34H SATISFIED WITH MAXIMUM RESIDUAL =, E15.6, 8H FOR W(I5,IH))
IF(ITOL.EQ.0) 673, 10
673 PRINT 9050
CALL PRIMAT(W)
GO TO 10
C
C TEST IF MAXIMUM NUMBER OF INNER-ITERATIONS EXCEEDED FOR VORTICITY.
675 IF(ITER6.GE.ITERMAX) 677, 505
677 PRINT 9677, RMAX6, ITER6
9677 FORMAT(//, 7H ****** MAXIMUM NUMBER OF ITERATIONS USED FOR W-ONLY
1 ITERATIONS, MAX RESIDUAL =, E12.4, 8H AT ITER, 16 )
MPN=1
PRINT 9009
CALL PRIMAT(PSI)
PRINT 9040
CALL PRTMAT(w)

C    END OF MAIN LOOP
70     CONTINUE
       STOP
       END
REFERENCES


11. J. Smith, "The coupled equation approach to the numerical solution of the biharmonic equation by finite differences," Report supported by NASA Sustaining University Program - NGR 43-001-021, Univ. of Tennessee, n.d.