NUMERICAL STUDIES OF TWO DIMENSIONAL, STEADY STATE NAVIER-STOKES EQUATIONS FOR ARBITRARY REYNOLDS NUMBER

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1. Introduction.

The development of the high speed digital computer has resulted in extensive efforts to solve numerically fluid problems whose equations of motion are the Navier-Stokes equations (see, e.g., references [1]-[5] and the additional references contained therein). The interest in these equations is founded not only on the fact that they incorporate boundary layer phenomena, but also on the important observation that they result from both microscopic and macroscopic approaches to viscous flow [6], [7].

In this paper we will discuss and illustrate a numerical method for two dimensional, steady state Navier-Stokes problems which applies with equal ease to cases of small Reynolds number and to cases of large Reynolds number. If and when such steady state flows exist, which is still usually an open matter, the method to be described is vastly more economical and accurate than time dependent, step-by-step methods. The power of the method is contained in the construction of difference equations which, for <u>all</u> R, yield diagonally dominant systems of algebraic equations.

2. Difference Analogues of the Steady State Navier-Stokes Equations.

In terms of the stream function ψ, the vorticity ω, and the Reynolds number R, the two dimensional, steady state Navier-Stokes equations are

$$(2.1) \qquad \Delta \psi = - \omega$$

$$(2.2) \qquad \Delta \omega + R \left(\frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x}\right) = 0 .$$

It will be convenient to approximate coupled equations (2.1) and (2.2) by linear elliptic difference equations, since the numerical solution of such equations is feasible in practice and well understood in theory [8].

For $h > 0$, then, consider the five points (x, y), $(x+h, y)$, $(x, y+h)$, $(x-h, y)$ and $(x, y-h)$, numbered 0, 1, 2, 3, 4, respectively, in Figure 2.1. For convenience, any function $u(x, y)$ defined at a point numbered i will be denoted at that point by u_i.

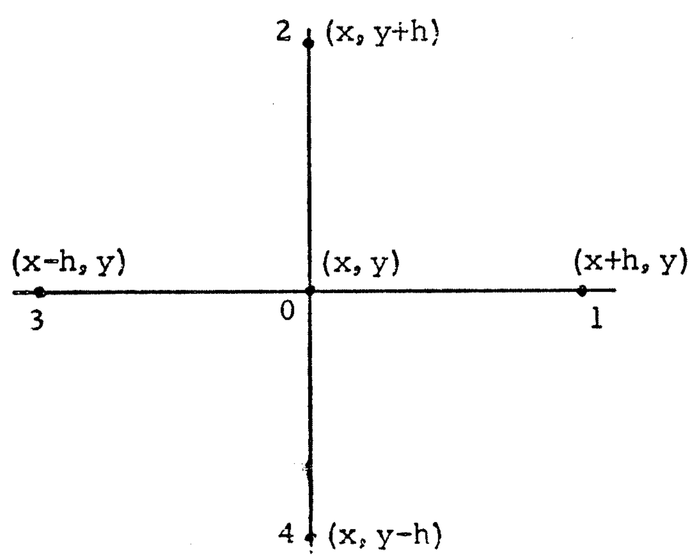

Figure 2.1

Suppose first that $\omega(x, y)$ is defined at the point numbered 0 in Figure 2.1. Then (2.1) will be approximated at 0 by the well known [8] Poisson difference analogue

(2.3) $$-4\psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 = -h^2 \omega_0 .$$

Next, suppose that $\psi(x, y)$ is defined at the points numbered 0, 1, 2, 3 and 4 in Figure 2.1. Then (2.2) can be approximated first by the difference-differential equation

(2.4) $$-4\omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 + h^2 R \left[\frac{\psi_1 - \psi_3}{2h} \frac{\partial\omega}{\partial y} - \frac{\psi_2 - \psi_4}{2h} \frac{\partial\omega}{\partial x}\right] = 0 .$$

For notational convenience, in (2.4) set

(2.5) $$A = \psi_1 - \psi_3$$

(2.6) $$B = \psi_2 - \psi_4 .$$

Then, to assure the dominance of the coefficient of ω_0 in (2.4), set

(2.7) $$\frac{\partial\omega}{\partial y} = \frac{\omega_2 - \omega_0}{h}, \qquad \text{if } A \geq 0$$

(2.8) $$\frac{\partial\omega}{\partial y} = \frac{\omega_0 - \omega_4}{h}, \qquad \text{if } A < 0$$

(2.9) $$\frac{\partial\omega}{\partial x} = \frac{\omega_0 - \omega_3}{h}, \qquad \text{if } B \geq 0$$

(2.10) $$\frac{\partial\omega}{\partial x} = \frac{\omega_1 - \omega_0}{h}, \qquad \text{if } B < 0 .$$

Thus, depending on the signs of A and B, substitution of (2.7) - (2.10) into (2.4) yields the following approximations of (2.2):

(2.11) $$(-4 - \frac{AR}{2} - \frac{BR}{2})\omega_0 + \omega_1 + (1 + \frac{AR}{2})\omega_2 + (1 + \frac{BR}{2})\omega_3 + \omega_4 = 0; \quad (A \geq 0, \ B \geq 0)$$

(2.12) $$(-4 - \frac{AR}{2} + \frac{BR}{2})\omega_0 + (1 - \frac{BR}{2})\omega_1 + (1 + \frac{AR}{2})\omega_2 + \omega_3 + \omega_4 = 0; \quad (A \geq 0, \ B < 0)$$

(2.13) $$(-4 + \frac{AR}{2} - \frac{BR}{2})\omega_0 + \omega_1 + \omega_2 + (1 + \frac{BR}{2})\omega_3 + (1 - \frac{AR}{2})\omega_4 = 0; \quad (A < 0, \ B \geq 0)$$

(2.14) $$(-4 + \frac{AR}{2} + \frac{BR}{2})\omega_0 + (1 - \frac{BR}{2})\omega_1 + \omega_2 + \omega_3 + (1 - \frac{AR}{2})\omega_4 = 0; \quad (A < 0, \ B < 0).$$

Generalizations of (2.3) and (2.11)-(2.14) to higher dimensions or for point arrangements different from that shown in Figure 2.1 follow readily [8], but will not be considered in this paper.

3. The Numerical Method.

Consider now fluid problems defined by (2.1) and (2.2) on a region $\mathcal{R}$ with boundary conditions prescribed on the boundary $\mathcal{R}'$ of $\mathcal{R}$. Since, in general, one does not know how to produce analytical solutions of such problems, we formulate next a numerical method for approximating such solutions.

For fixed $h > 0$, construct a set of interior grid points $\mathcal{R}_h$ and a set of boundary grid point $\mathcal{R}_h'$ in the usual way [8]. Define an initial vector $\omega^{(0)}$ on $\mathcal{R}_h$ and then apply (2.3) at each point of $\mathcal{R}_h$ to yield a system of linear algebraic equations. Solve this system by successive over-relaxation and denote its solution by $\psi^{(1)}$. Using these values of $\psi^{(1)}$, apply (2.11)-(2.14) at each point of $\mathcal{R}_h$ to yield a linear algebraic system. Solve this system by over-relaxation and denote the solution by $\omega^{(1)}$. Next, in the fashion indicated above, use (2.3) and $\omega^{(1)}$ to generate $\psi^{(2)}$, and then use (2.11)-(2.14) and $\psi^{(2)}$ to generate $\omega^{(2)}$. Continue, in this fashion, to generate $\psi^{(k+1)}$ from (2.3) and $\omega^{(k)}$ and to generate $\omega^{(k+1)}$

from (2.11)-(2.14) and $\psi^{(k+1)}$. Terminate the iteration if and when $\psi^{(m+1)} = \psi^{(m)}$ for some value m. The terminal vectors $\psi^{(m)}$ and $\omega^{(m)}$ are defined to be the numerical approximations of $\psi(x,y)$ and $\omega(x,y)$, respectively, on $\mathcal{R}_h$.

4. Examples.

In order to illustrate the numerical method described in Section 3, we will consider an example of the cavity flow of a viscous, incompressible fluid. Such flows are of interest in studying the von Karman vortex street [2] and in studying the flow through a grooved channel [9]. The problem will be formulated as follows.

Let $\mathcal{R}'$ be the unit square (consult Figure 4.1) whose vertices are (0,0), (1,0), (1,1) and (0,1). Let $\mathcal{R}$ be the interior of $\mathcal{R}'$. Consider a fluid in $\mathcal{R}$ whose motion is defined by (2.1), (2.2) and the following boundary conditions:

(4.1) $$\psi = 0 \quad , \text{ on } \mathcal{R}'$$

(4.2) $$\omega = 1 \quad , \text{ on OC and CB}$$

(4.3) $$\omega = -1 \quad , \text{ on OA and AB .}$$

Of the variety of solutions generated on the CDC 3600 by the method of Section 3, only results for the three typical ones defined by $h = .02$, $R = 10$, $R = 1000$, and $R = 3000$ are shown in Figures 4.2-4.4. The respective running times were 15 min 50 sec (for iterations converging to 10^{-8}), 45 min 35 sec (for iterations converging to 10^{-8}), and 26 min 10 sec (for iterations converging to 10^{-6}). The over-relation factor 1.8 was used to solve for each $\psi^{(k)}$, while 1.3 was used to solve for each $\omega^{(k)}$. The initial vector $\omega^{(0)}$ was defined in each case to be the zero vector. The maximum values of ψ for $R = 10$, 1000 and 3000 were 0.0185, 0.0159, and 0.0130, respectively, and these occurred at (.26, .74),

(.26, .66) and (.28, .66), respectively. Because both ψ and ω are skew symmetric with respect to the 45° line, we have graphed ψ only above the 45° line and ω only below the 45° line in each of Figures 4.2-4.4.

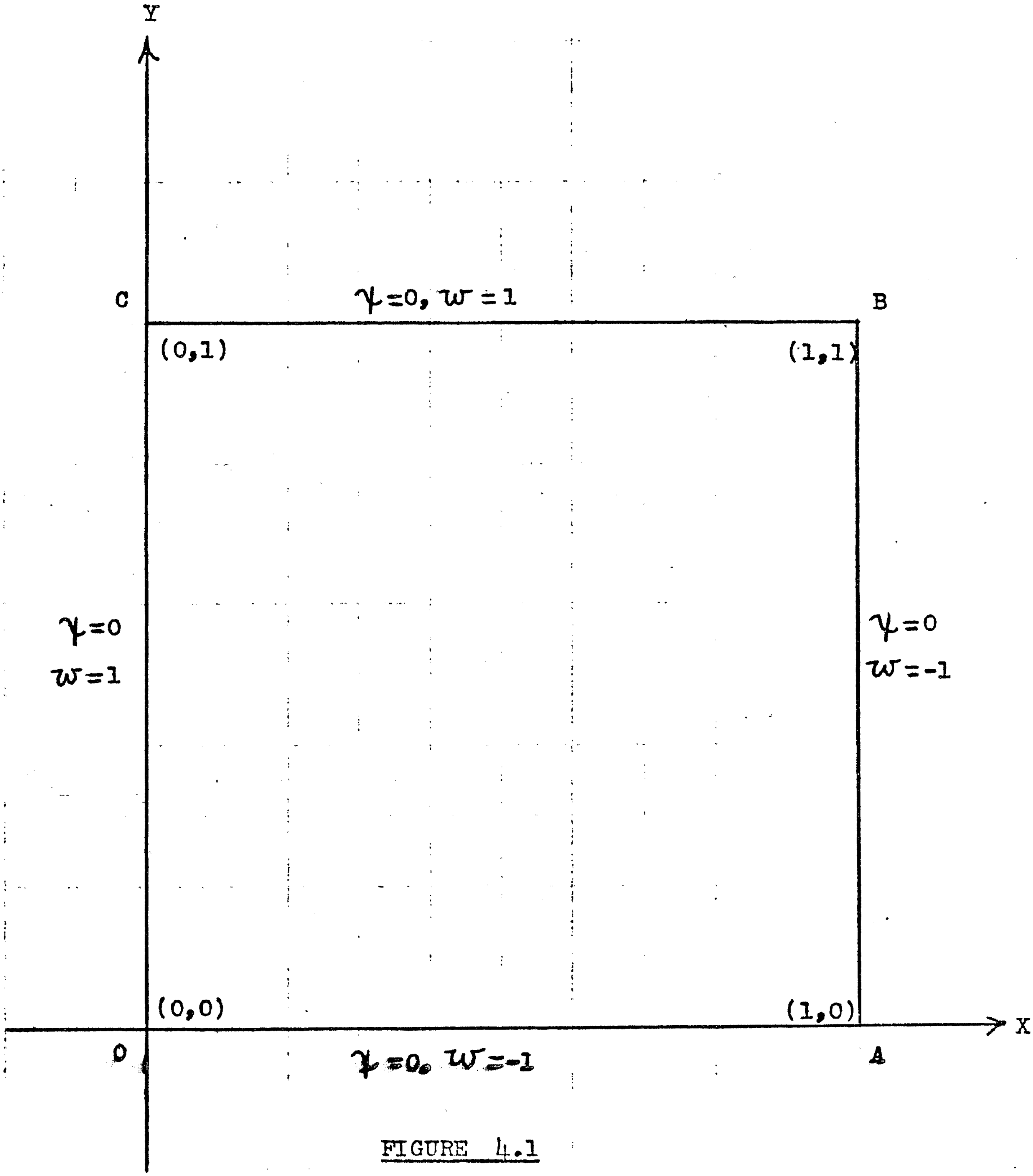

FIGURE 4.1

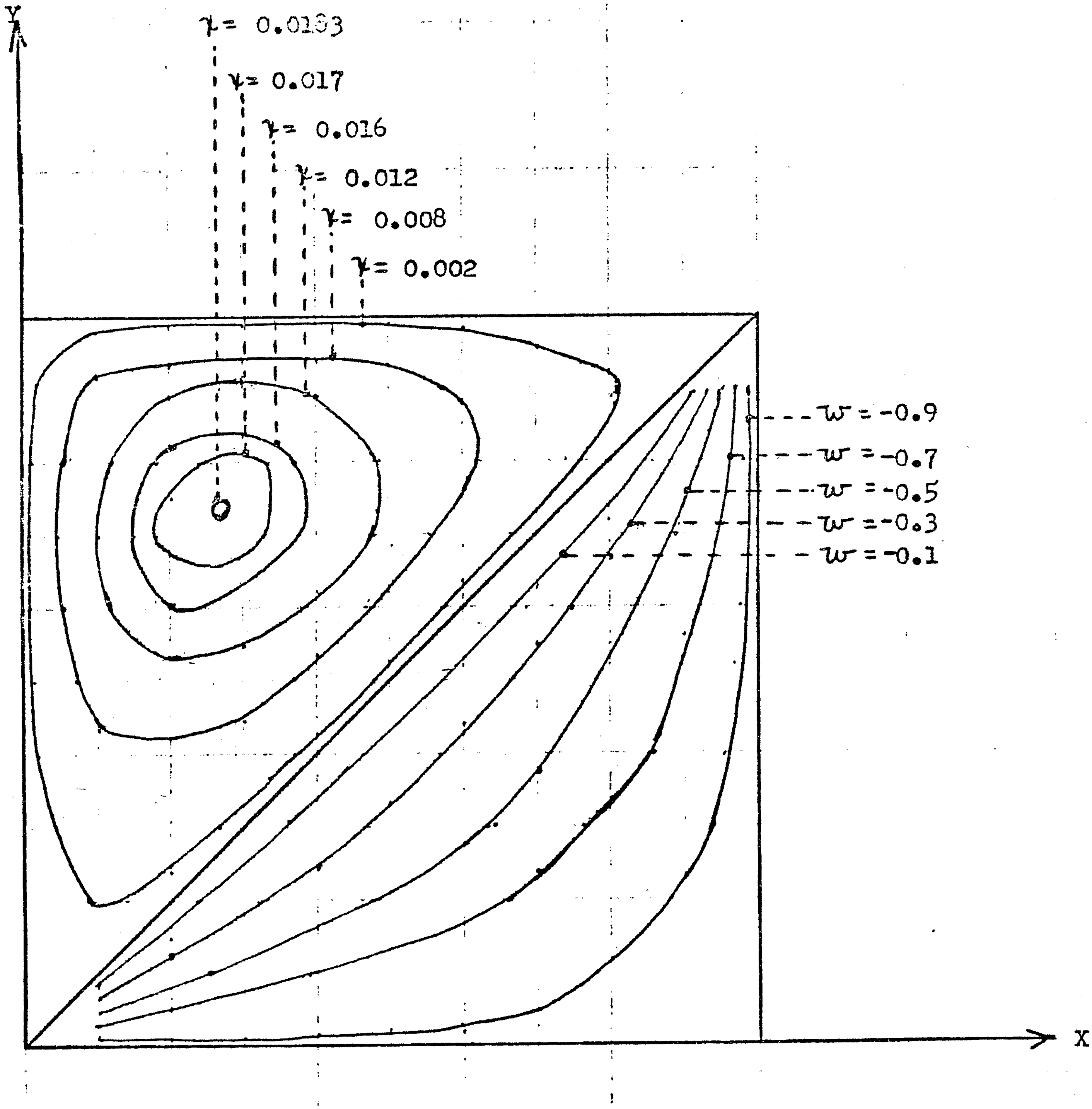

FIGURE 4.2 R = 10.

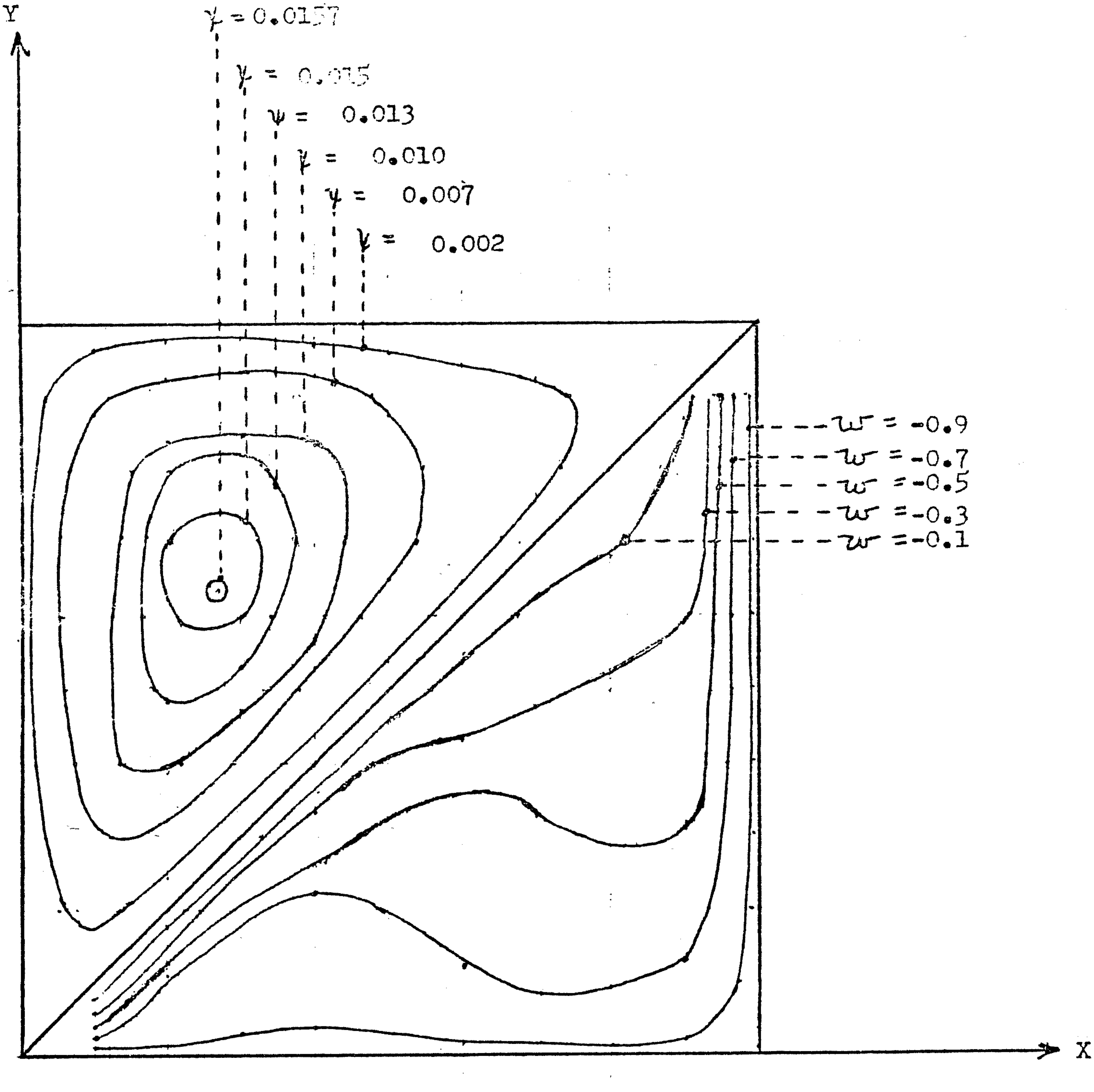

FIGURE 4.3 R = 1000.

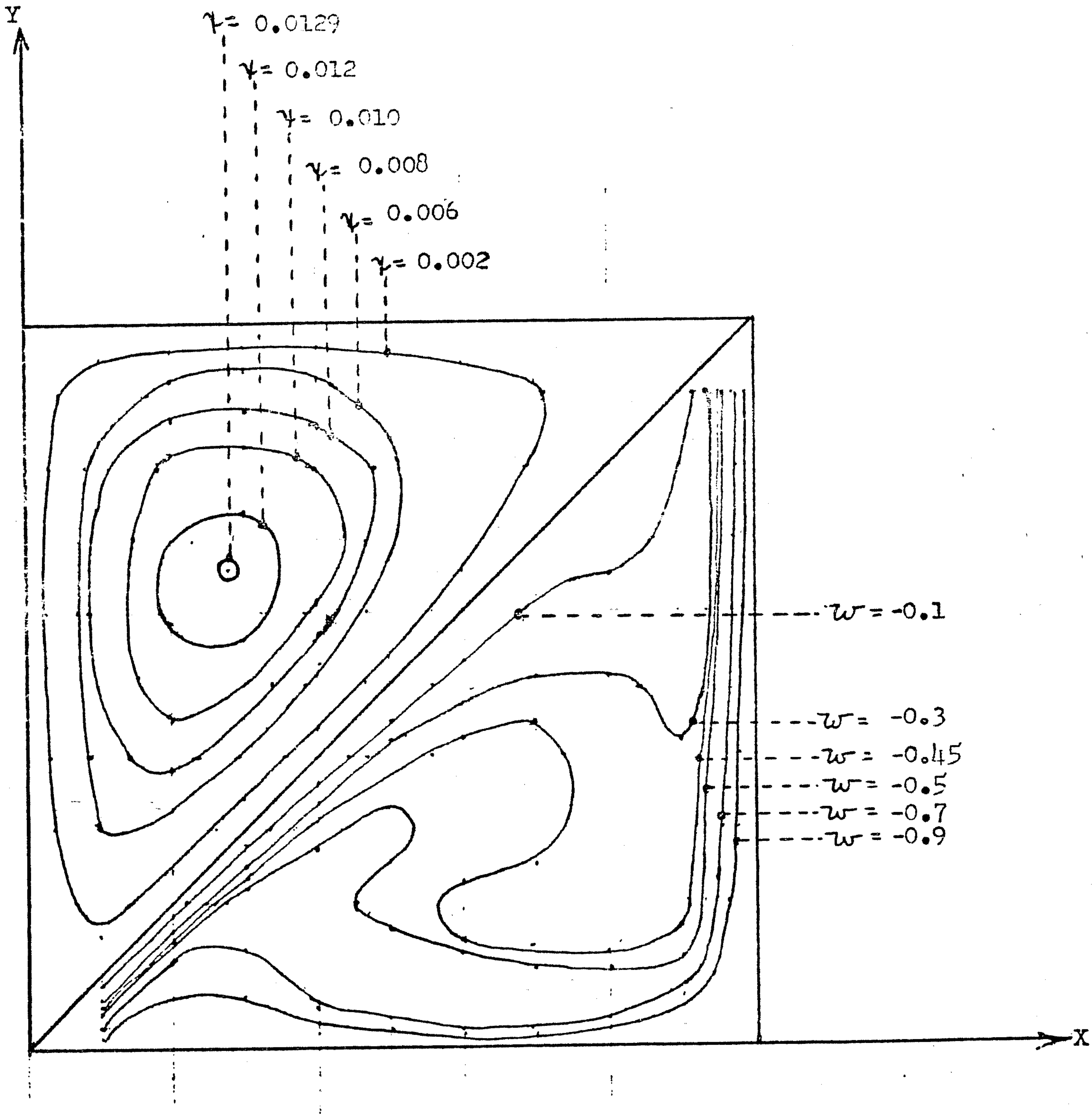

FIGURE 4.4 R = 3000 .

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