Character Table Isomorphisms

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• Consider the equation $x^2 = y$, where $y \in \mathbb{R}$. We know there are two solutions $x$ and $-x$. 
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• If we write $\sqrt{9}$ we really mean 3.

• What about $\sqrt{-1}$?
• Groups give us a language to talk about basic properties of “isomorphism” and equivalence.

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Given an object $T$ we can talk about equivalent objects to $T$. 
• Associated to each finite group is an object called a character table.

• The characters are the shadows of the group.

$$\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
2 & 0 & \zeta + \zeta^4 & \zeta^2 + \zeta^3 \\
2 & 0 & \zeta^2 + \zeta^3 & \zeta + \zeta^4
\end{pmatrix},$$

$$\zeta^5 = 1, \text{ i.e., } \zeta = e^{(2\pi i/5)}.$$
Shadows
Building a Database

- To help us understand what information about a group $G$ is recoverable from its character table, we are building a database of small finite groups with the same character tables.

- We want to compare about 450,000,000 character tables.
Comparing Two Tables

• The character table of a group $G$ has no canonical ordering, i.e., there is no canonical way of picking which column or row appears where.

• Given two $n$-by-$n$ character tables $M$ and $N$. We say $M = N$ if some permutation of the row and columns of $M$ equals the table $N$. 
Comparing Two Tables

\[
\begin{pmatrix}
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 \\
2 & -1 & 1 & -1 \\
-2 & 1 & -1 & 1 \\
\end{pmatrix}
= ?
\begin{pmatrix}
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 \\
-2 & -1 & 1 & -1 \\
2 & 1 & -1 & 1 \\
\end{pmatrix}
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\]

- We encode the table as a graph and run graph isomorphism.
• Consider the table:

\[
\begin{pmatrix}
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2 & -1 & 1 & -1 \\
-2 & 1 & -1 & 1
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\]
Encoding as a Graph

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Method

- We run an initial hash:
  - Given a group →
  - Construct table →
  - Create hash.
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\]

- The hash is the multiset of rows, where each is a multiset.

\[
\{\{-1^2, 1^2\}^2, \{-1^2, 1, 2\}, \{-2, -1, 1^2\}\}.
\]
• We run an initial hash:
  • Given a group →
  • Construct Table →
  • Create Hash.

• SmallGroup(512, 64889569) gives
  2dff0c4ba891481cd4fa6e2dc65f298c.

• SmallGroup(512, 64889570) gives
  cd246c40463c53d07d13052186170424.

• SmallGroup(512, 54890438) gives
  2dff0c4ba891481cd4fa6e2dc65f298c.
Method

- For each hash bucket run an all against all.
- Each bucket is mostly a single job.
Results

Computing row-equivalence classes

SmallGroups of order 1535

Execution time (hours)

Percent of Total

0 5 10 15 20
Results

Size of Row-equivalent classes

Small groups of order 1536

Classes with <= 1000 groups

Number of groups

Number of classes

0

500000

1000000

0 200 400 600 800 1000

Small groups of order 1536
Classes with <= 1000 groups
### Results

#### Size of Row-equivalent classes

<table>
<thead>
<tr>
<th>Number of classes</th>
<th>Number of groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50000</td>
</tr>
<tr>
<td></td>
<td>100000</td>
</tr>
<tr>
<td></td>
<td>150000</td>
</tr>
<tr>
<td></td>
<td>200000</td>
</tr>
<tr>
<td></td>
<td>250000</td>
</tr>
</tbody>
</table>

#### Graph

- **Title:** Size of Row-equivalent classes
- **X-axis:** Number of classes
- **Y-axis:** Number of groups
- **Legend:**
  - Small groups of order 1536
  - Classes with > 10000 groups

The graph shows the distribution of row-equivalent classes for small groups of order 1536 and classes with more than 10,000 groups.
• We are grateful to the GAP and HTC-Condor community for project support and troubleshooting.
• Thank you for your time.