A MW Framework for Solving Recursive Economic Models

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Wealth Accumulation Example

Given initial capital stock $x_0$, find $V(x_0)$

$$V(x_0) = \begin{cases} \max_{(c_t,l_t)} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ s.t. \quad x_{t+1} = x_t + f(x_t, l_t) - c_t \end{cases}$$

- $c_t$ and $l_t$ are consumption and labor supply at time $t$
- capital evolves according to $x_{t+1} = x_t + f(x_t, l_t) - c_t$
- $\beta$ is the discount factor and $u(c_t, l_t)$ is the utility given consumption $c_t$ and labor supply $l_t$
- $V(x)$ is the value function for $x_0 = x$
As written, this is an optimization problem with infinitely many variables: \( c_t, l_t, x_t, t = 0, 1, 2, \ldots \), so as written it is impossible to solve. But we can make use of the *dynamic programming principle*, based on the observation that the optimal objective \( V(x_0) \) depends only on \( x_0 \). Also note by the form of the objective and constraints, we have at the optimal values of \( x_t, c_t, l_t \) that

\[
V(x_0) = u(c_0, l_0) + \beta \sum_{t=0}^{\infty} \beta^t u(c_{t+1}, l_{t+1})
\]

\[
= u(c_0, l_0) + \beta V(x_1)
\]

\[
= u(c_0, l_0) + \beta V(x_0 + f(x_0, l_0) - c_0).
\]

We can use this formula involving \( V \) to solve for the entire function \( V \), not just find its value at a given \( x_0 \).
Bellman Equation for $V(x)$

\[
V(x) = \max_{(c,l)} u(c, l) + \beta V(x + f(x, l) - c) \tag{1}
\]

- The function $V$ is unknown
- Parametric dynamic programming: Approximate $V(x)$ by $\hat{V}(x; a)$, and solve for the parameters $a$ using the information in the Bellman equation.
  - e.g., $\hat{V}(x; a) = \sum_{j=0}^{p} a_j x^j$
  - find $a \in \mathbb{R}^p$ such that $\hat{V}(x; a)$ “approximately” satisfies the Bellman equation (1), on a finite grid of $x$ values: $x_1, x_2, \ldots, x_n$. 

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Value Function Iteration

Step 0. **Initialization.** Choose functional form for $\hat{V}(x; a)$ and approximation grid $X = \{x_1, \ldots, x_n\}$.
Make initial guess $\hat{V}(x; a^0)$ and choose $\epsilon > 0$.

Step 1. **Maximization step.** Fix $a^k = (a^k_j)_{j=1}^p$.
For $i = 1, \ldots, n$, compute

$$v_i = T\hat{V}^k(x_i, a^k) = \max_{(c_i, l_i)} u(c_i, l_i) + \beta \hat{V}(x^+_i, a^k)$$

where $x^+_i = x_i + f(x_i, l_i) - c_i$

Step 2. **Fitting step.** Fix $c, l$. Find $a^{k+1}$ s.t.

$$a^{k+1} = \arg\min_a \|\hat{V}(x, a) - v\|^2$$

Step 3. **Convergence.** If $\|\hat{V}(x, a^{k+1}) - \hat{V}(x, a^k)\|_\infty > \epsilon$, go to Step 1; otherwise stop and report solution.
Value Function Iteration in MW

Objective: Solve the Bellman equation (1)

**MASTER:** Initialization. Choose functional form for $\hat{V}(x; a)$ and approximation grid $X = \{x_1, \ldots, x_n\}$. Make initial guess $\hat{V}(x; a^0)$ and choose $\epsilon > 0$.

**WORKER:** Maximization step. Fix $a^k = (a^k_j)_{j=1}^p$. For $i = 1, \ldots, n$, compute (in parallel)

$$v_i = T\hat{V}^k(x_i, a^k) = \max_{(c_i, l_i)} u(c_i, l_i) + \beta \hat{V}(x_i^+, a^k)$$

**MASTER:** Fitting step. Fix $c, l$. Find $a^{k+1}$ s.t.

$$a^{k+1} = \arg\min_a \|\hat{V}(x, a) - v\|^2$$

**MASTER:** Convergence. If $\|\hat{V}(x, a^{k+1}) - \hat{V}(x, a^k)\|_\infty > \epsilon$, go to Step 1; otherwise stop and report solution.
Each task finds the optimal \((c_i, l_i)\) for a batch of \(x_i\)'s.

- Calls a FORTRAN code (to demonstrate that we can!) to do minimizations.
- Hot starting - the optimal \((c_i, l_i)\) is usually a great starting point for \((c_{i+1}, l_{i+1})\).
- The task “wrapper” and the FORTRAN code communicate via files.
MW Implementation Notes

- `act_on_complete_task()` on the Master stores the $v_i$’s as they arrive from the workers. When all workers have reported, it solves the least-squares problem (fitting step) to find $a_{k+1}$.
  - Could still take a fitting step without waiting for all tasks to report, to avoid hangups if some workers go down.
  - Could adapt size of task (number of $x_i$’s in the batch) to accommodate workers of different speeds.
How Big Can These Get?

Judd: These models can get very Big!!!

- **Investment Portfolio**
  - $d$ assets in the portfolio
  - $X_j = \{x_{j1}, \ldots, x_{jn}\}$ represents $j$-th asset’s position
  - state space: $X = X_1 \times X_2 \times \cdots \times X_d$
  - transaction cost occurs when adjusting asset positions

- **Dynamic Principal-Agent Problem**
  - the CEO’s performance is evaluated by multiple measures, e.g. stock price, annual profits, etc.
  - the company decides the CEO’s compensation package

- **Many other economic applications**
Current and Future Work

• Adapting a Fortran90 code with a more complex representation for $\hat{V}(x, a)$, multidimensional $x$, “legacy” minimization routine.

• Dynamic Programming is
  ○ ubiquitous
  ○ compute-intensive,
  ○ algorithmically well suited to the master-worker paradigm supported by MW.

• We are investigating a suitable abstracted implementation of dynamic programming in MW: MW-DP.