Pinhole Camera Model

- Aka perspective projection model

- For each scene point only 1 ray can enter camera.
- Pinhole = center of projection through which all light passes.
- Pinhole too big => blurring.
- Pinhole too small => diffraction-based blurring.
- Long exposure time needed.

Perspective Projection

- Image plane orthogonal to z axis (called optical axis).
- Camera frame origin at center of projection.
- 3D scene point $\mathbf{I} = (X,Y,Z)^T$ projects to image point $\mathbf{p} = (x,y)^T$ where $z' = \frac{f}{z}$ (focal length).
Before the Discovery of Perspective

di Bartolo, “The Nativity of the Virgin” (c. 1400)

di Giovanni Fei, “The Presentation of the Virgin” (c. 1400)

Ambrogio Lorenzetti (1342) The presentation in the temple. Panel, Uffizi, Florence
Natural Perspective

Euclid’s Optics (300 BC)
- Visual ray: from point on object to eye
- Visual cone: from contour of object to eye
- Euclid’s Law: diminution in visual angle with distance

Italian Renaissance
- Linear perspective
  - Illusionistic 3D space
  - Sculptural body
  - Natural pose, individual expression
  - Humanized suffering
“Perspective is nothing else than the seeing of an object through a sheet of glass, on the surface of which may be marked all the things that are behind the glass.”

-- Leonardo

Hieronymous Rodeem (1531) Johan II of Bavaria. Woodcut.

Alberti’s Window

“First of all, on the surface on which I am going to paint, I draw a rectangle of whatever size I want, which I regard as an open window, through which the subject to be painted is seen.”

-- Alberti (1435-6)
Alberti’s Veil
• Grid system

Albrecht Dürer (c. 1525) *Draughtsman drawing a reclining nude.* Woodcut.

Point-Plotting Method
• Use strings to embody Euclid’s visual rays

Albrecht Dürer (c. 1525) *Two draughtsmen plotting points for the drawing of a lute in foreshortening.* Woodcut.
Alberti’s Method (1435): “Construzione Legittima”

1. Draw “open window”, with a human figure 3 braccia high
2. Mark baseline in units of 1 braccio
3. Draw Centric Point at eye level (determines severity of convergence)
4. Draw orthogonals
5. Draw horizon line

6. Draw “little space”, with a point at the height of the Centric Point
   (like elevation view, with eye point)
7. Draw baseline with units of 1 braccio (like ground plane)
8. Draw a vertical line (like picture plane)
9. Draw diagonals (like visual rays)
10. Draw transversals at intersections
Modified Alberti Method

• Slide the “little space” over so the right side of the rectangle becomes the picture plane
• DB is a “check line” for verifying correctness

Masaccio’s “Trinity” (c. 1425-8)

• The oldest existing example of linear perspective in Western art
• Use of “snapped” rope lines in plaster
• Vanishing point below orthogonals implies looking up at vaulted ceiling
Piero della Francesca, “Flagellation of Christ” (c. 1455)

- Carefully planned
- Strong sense of space
- Low eye level

Leonardo da Vinci, “Last Supper” (c. 1497)

- Use of perspective to direct viewer’s eye
- Strong perspective lines to corners of image
Properties of Perspective Projection

- Object size changes as it translates along $z$ axis (scale effect)

\[ \text{Magnification } m = \left( \frac{d}{d'} \right) = \left( \frac{f}{z_0} \right) \]

$\Rightarrow$ distance b/w points not preserved

- As $f$ gets smaller, more world points project onto finite image plane $\Rightarrow$ more wide angle image

- As $f$ gets larger, more telescopic

- Lines in 3D project to lines in 2D

Distant Objects are Smaller
Parallel Lines Meet

Common to draw film plane in front of the focal point.

Moving the film plane merely scales the image.

Geometric Properties of Projection

- Points go to points
- Lines go to lines
- Planes go to whole image
- Polygons go to polygons
- Degenerate cases
  - line through focal point to point
  - plane through focal point to line
Perspective Projection (cont.)

- Perspective projection equations

\[ \frac{P'}{P} = \frac{z'}{z} \]

\[ \begin{align*}
    x' &= \frac{fx}{z} \\
    y' &= \frac{fy}{z} \\
    z' &= f
\end{align*} \]

- Vanishing point = point in image beyond which projection of straight line cannot extend

- Focus of Expansion (FOE)
  When camera translates, trajectories of image points appear to move towards or away from a fixed point called FOE which is common point called vanishing point because all points move along straight lines relative to camera
Vanishing Points

- each set of parallel lines (= direction) meets at a different point
  - The vanishing point for this direction
- Sets of parallel lines on the same plane lead to *collinear* vanishing points
  - The line is called the horizon for that plane
- Good ways to spot faked images
  - scale and perspective don’t work
  - vanishing points behave badly
  - supermarket tabloids are a great source

Using Homogeneous Coordinates

- Translation by \((a, b, c)\)
  \[
  \begin{align*}
  x' &= x - a \\
  y' &= y - b \\
  z' &= z - c
  \end{align*}
  \]
  \[
  \begin{bmatrix}
  x' \\
  y' \\
  z' \\
  1
  \end{bmatrix} =
  \begin{bmatrix}
  1 & 0 & 0 & a \\
  0 & 1 & 0 & b \\
  0 & 0 & 1 & c \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y \\
  z \\
  1
  \end{bmatrix}
  \]
  \[P' = TP\]
- Scale Change by \((s_x, s_y, s_z)\)
  \[
  \begin{align*}
  x' &= s_x x \\
  y' &= s_y y \\
  z' &= s_z z
  \end{align*}
  \]
  \[
  \begin{bmatrix}
  x' \\
  y' \\
  z' \\
  1
  \end{bmatrix} =
  \begin{bmatrix}
  s_x & 0 & 0 & 0 \\
  0 & s_y & 0 & 0 \\
  0 & 0 & s_z & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y \\
  z \\
  1
  \end{bmatrix}
  \]
  \[P' = SP\] where
  \[
  S =
  \begin{bmatrix}
  s_x & 0 & 0 & 0 \\
  0 & s_y & 0 & 0 \\
  0 & 0 & s_z & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \]
Rotation about coordinate axes

(Counterclockwise looking towards origin)

Ex. About \( z \)-axis:

\[
\begin{align*}
\xi' &= \xi \cos \theta - \eta \sin \theta \\
\eta' &= \xi \sin \theta + \eta \cos \theta \\
\zeta' &= \zeta
\end{align*}
\]

\[
R_z = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Also,

\[
R_x = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

\[
R_y = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

Any transformation involving translation, scale, or rotation can be written as:

\[ P = MP \]

where \( M \) constructed by composing transformation matrices

Ex.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\gamma & 0 & 0 \\
0 & \gamma & 0 \\
0 & 0 & \gamma
\end{bmatrix}
\begin{bmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\alpha \cos \theta & \beta \sin \theta & \alpha (\gamma \cos \theta - \beta \sin \theta) \\
-\beta \cos \theta & \alpha \sin \theta + \beta \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Translations are commutative, rotations are not.

General transformation matrix of form:

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & T_x \\
a_{21} & a_{22} & a_{23} & T_y \\
a_{31} & a_{32} & a_{33} & T_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Camera Matrix

- Turn previous expression into HC’s
  - HC’s for 3D point are (X,Y,Z,T)
  - HC’s for point in image are (U,V,W)
Projection Matrix for Orthographic Projection

\[
\begin{bmatrix}
U \\
V \\
W \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z \\
T \\
\end{bmatrix}
\]

*Note: Since image plane at t = f, perspective projection equation can be written as:

\[
\frac{X}{Z} = \frac{f}{0}\begin{bmatrix}X \\
Y \\
Z \\
\end{bmatrix}
\]

and:

\[
\begin{cases}
X' = X/Z \\
Y' = Y/Z
\end{cases}
\]

\(\Rightarrow\) Camera = linear projective transform from 3D projective space to 2D projective plane.

* 3x4 matrix called camera perspective projection matrix
Camera Parameters

- Issue
  - camera may not be at the origin, looking down the z-axis
    - extrinsic parameters
  - one unit in camera coordinates may not be the same as one unit in world coordinates
    - intrinsic parameters - focal length, principal point, aspect ratio, angle between axes, etc.

\[
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix}
= \begin{bmatrix}
\text{Transformation} & 1 & 0 & 0 \\
\text{representing} & 0 & 1 & 0 \\
\text{intrinsic parameters} & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\text{Transformation} & 0 & 0 & 0 \\
\text{representing} & 1 & 0 & 0 \\
\text{extrinsic parameters} & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z \\
T
\end{bmatrix}
\]