CS559 – Lecture 24 (part 1)
Cubic Derivations

These are course notes (not used as slides)
Written by Mike Gleicher, Oct. 2005
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General Polynomial forms

- f(u) = u a (u = [u^0 u^1 u^2 …])
- f(u) = u B p
- Bezier Cubic:
  - Note – we did uBp as book

$$\begin{matrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{matrix}$$

Blending functions

$b_0(u) = 1\text{-}3u+3u^2\text{-}u^3 = 1\,(1\text{-}u)^3\,u^0$

$b_1(u) = 3u\text{-}6u^2+3u^3 = 3\,(1\text{-}u)^2\,u^1$

Different Matrix, Different Spline

- Cardinal Spline Matrix (f = u B p)
  - S = (1-t)/2 (s=1/2 for Catmull Rom)

$$\begin{matrix} 0 & 1 & 0 & 0 \\ -s & 0 & s & 0 \\ 2s & s-3 & 3-2s & -s \\ -s & 2-s & s-2 & s \end{matrix}$$

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b0 =  -s u +   2 s u² –      s u³
b1 = 1      + (s-3) u² + (2-s) u³
b2 =   s u + (3-2s)u² + (s-2) u³
b3 =                -s u² +    s u³

Check: (u=0) [0,1,0,0]
          (u=1) [0,0,1,0]
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How do you derive these?

- Let's consider the line segment
  - (1-u) p0 + u p1 vs. a0 + a1 u
- Constraints
  - P0 = f(0) = a0 + a1 0 = a0
  - P1 = f(1) = a0 + a1 1 = a0 + a1
- Linear System (to solve simultaneously)
  - p = C a C = [1 0 ; 1 1]
- Solve Linear System
  - a = C^{-1} p B = C-1 = [1 0; -1 1]

Matrix form

- Can write canonical form as **a u** **u**=[u^0 u^1 u^2 ⋯]
- Given definitions of p, solve for **a** in terms of **p**
  - Line segment example

$f(0) = p_0$

$f(1) = p_1$

Plug in canonical equations

$p_0 = a_0\,0^0 + a_1\,0^1$

$p_1 = a_0\,1^0 + a_1\,1^1$

Matrix form

p = C a

Where

C =

1	0
1	1

Basis Matrices

- The matrix C is called the constraint matrix
- The inverse of C is called B, the Basis Matrix
  - **a = B p**
- Since **f**(u) = **u ¢ a**
  - **f**(u) = **u B p** - **u B** are the blending functions
- In the example, **C**= [1 0 \ 0 1] **B= C-1**
  - B =

1	0
-1	1

$b_0(u) = 1\text{-}u$

$b_1(u) = u$

More complicated example: Catmull Rom Splines

- $f(0) = p_1$
- $f(1) = p_2$
- $f'(0) = ½ (p_2 – p_0)$
- $f'(1) = ½ (p_3 – p_1)$

- $p_0 = f(1) – 2 f'(0)$
- $p_3 = f(0) + 2 f'(1)$

- Remember…

$f(u) = a_0 + a_1 u^1 + a_2 u^2 + a_3 u^3$

- So

$f'(u) = a_1 + 2 a_2 u + 3 a_3 u^2$

1	1 – 2	1	1
1			
1	1	1	1
1	2 (1)	2(2)	2(3)

Catmull Rom Blending Function

- $B = C^{-1}$

0	1	0	0
- ½	0	½	0
1	-2 ½	2	- ½
- ½	1 ½	-1 ½	½

$b_0(u) = - ½u + u^2 - ½u^3$
$b_1(u) = 1 - 2 ½ u^2 + 1 ½ u^3$
$b_2(u) = ½ u – 2 u^2 - 1 ½ u^3$
$b_3(u) = - ½ u^2 + ½ u^3$

Can you do this for Bezier Cubics?

- Yes – set up constraints and solve
  - f(0) = p0
  - f(1) = p3
  - f'(0) = 3 (p1 – p0)
  - f'(1) = 3 (p3 – p2)
- Doesn't generalize well (OK for 2,3,4)
- General form for Bezier blending functions
  - Bernstein Basis Polynomials