CS559 – Lecture 21/22
Curves

These are course notes (not used as slides)
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Shape Modeling

- Creating Mathematical Descriptions of Shape
- Why?
  - Drawing, Sample, Analyze
- Why is this hard
  - Shapes can be arbitrary and complex – hard to describe
  - Conflicting goals
    - Concise
    - Intuitive
    - Expressive
    - Analyzable
    - …

What is a Shape

- Mathematical definition is elusive
- Set of Points
  - Potentially (usually) infinite
- "Lives" in some bigger space (e.g. 2D or 3D)
- Many ways to describe sets
  - Set inclusion test (implicit representation)
  - Procedural for generating elements of the set
  - Explicit mapping from a known set

Some kinds of Shapes

- Curves
  - 1D Objects, like what you draw with a pen
- Surfaces / Areas
  - 2D Objects – the insides of 2D things
  - Bounded by a Curve
- Solids / Volumes
  - 3D Objects – the insides of things that take up volume
  - Different definition: set with the same dimension as the embedded space (an area of 2D)

Curves

- Intuitively, something you can draw with a pen
  - Not filled areas
  - Mathematical oddity: space filling curves
    - Requires infinite lengths, …
- Almost every point has 2 "neighbors"
- Locally equivalent to a line

Defining Curves

- Two different mathematical definitions
  1. The continuous image of some interval
  2. A continuous map from a one-dimensional space to an n-dimensional space
- Both definitions imply a mapping
  - From a line segment (which is a curve)
- #1 is a set of points, #2 is the mapping

Describing Curves

- Some curves have names
  - Line, line segment, ellipse, parabola, circular arc
- Some set of parameters to specify
  - Radius of an arc, endpoints of a line, …
- Other curves do not have distinct names
  - Need a *Free Form* representation

Curve Represenations

- Implicit
  - Function to test set membership
  - F(x,y) = 0
- Explicit / Parametric
  - Y = f(x)
  - (x,y) = f(t) – where t is a free parameter
    - Need to define a range for the parameter
- Procedural
  - Some other process for generating points in the set
- By definition, a curve has at least 1 parametric representation

Parameterizations

- For any curve (set of points) there may be many mappings from a segment of the reals
- Consider: line from 0,0 -> 1,1
  - (x,y) = (t,t)   t in [0,1]
  - (x,y) = (.5t, .5t)   t in [0,2]
  - (x,y) = (t^2,t^2)   t in [0,1]
- Many ways to represent a curve
- Are they the "same" (same set of points, diff parameterizations – so depends on defn of curve)

Free Parameters

- Not really a property of the curve
  - Many different parameterizations
- Think of it as time in the pen analogy
  - Parameterization says "where is pen at time T"
  - Many different ways to trace out the same curve have different timings
- Can "reparameterize" a curve
  - Same curve, different parameterization
  - Add a function f(t) -> f(g(t))   g 2 R->R

Some nice Parameterizations

- Unit Parameterization
  - Parameter goes from 0 to 1
  - No need to remember what the range is!
- Arc-Length Parameterization
  - Constant magnitude of 1st derivative
  - Constant rate of free parameter change = constant velocity
  - Arc-length parameterizations are tricky

How do we define functions?

- Simple shapes: easy
- Complex shapes, divide and conquer
  - Break into small pieces, each an easy piece
  - Approximate if needed
    - Add more pieces to get better approximations
  - Need to make sure pieces connect
- Typically, pick simple, uniform pieces
  - Line segments, polynomials, …

Parametric Values for Compound Curves

- Could reparameterize however we want
- One parameter space for all pieces
- Switching at various points
- KNOTS are the switching points
  - (0, .5, 1) in the case below

$$f(u) = \begin{cases} f_1(2*u) & \text{if } 0 \le u < \frac{1}{2} \\ f_2(2*u-1) & \text{if } \frac{1}{2} \le u < 1 \end{cases}$$

Connecting Pieces

- Only concerned about the knots
  - Assume the pieces are smooth
- Connection & Smoothness
  - Connection is a type of smoothness
- Derivative continuity
  - 0^{th} derivative = position
  - 1^{st} derivative = direction
  - 2^{nd} derivative = curvature

Types of Continuity

- C(n) continuity
  - Derivatives up to (and including N) match
  - May have less meaning since parameterizations don't mean anything
- G(n) continuity
  - C(0)
  - Higher derivatives may differ by a scale factor
  - Technically – c(n) in arc-length parameters
- How smooth?
  - C(2) = smooth in graphics
  - Higher continuity in design (boat hulls, …)

What kinds of pieces?

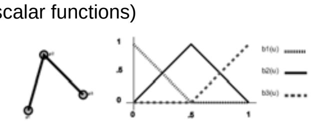
- Line segments
- Low-order polynomials
  - Quadrics (degree 2)
  - Cubics (degree 3)
  - Quartics, quintics, …
- Cubics are most popular in graphics
  - Best balance
- One polynomial per dimension
  - Or, coefficients are vectors (but free parameter is scalar)

What to control

- Control points
  - Where a curve goes (at a particular parameter value)
  - Derivaive (at a particular parameter value)
  - Interpolation vs. influence
- Specify values at a site
- Specify line segment
  - End points
  - Center and one end
  - Center and offset to end
  - Center, length, orientation (non-linear change)

Line segments

- Endpoints $\mathbf{p_1}$ and $\mathbf{p_2}$
  - $\mathbf{p} = (1-u)\,\mathbf{p_1} + u\,\mathbf{p_2}$
- Blending functions
  - $\mathbf{p} = b_1(u)\ \mathbf{p_1} + b_2(u)\ \mathbf{p_2}$
  - Convenient way to describe functions (including polynomials)
  - Basis functions (scalar functions)



Line Segment Bases

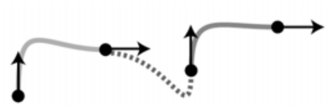
- Could choose different controls for line segment
  - Whatever was convenient
- Find conversions between different representations

Cubics

- Different than book: explain cubic forms first, derive them second (or maybe not at all)
- Canonical form for polynomial
  - $f(u) = \sum a_i u^i$
  - Vector a of coefficients
- Polynomial coefficients not very convenient
  - $a_3 u^3 + a_2 u^2 + a_1 u + a_0$
- One polynomial per dimenstion

Different ways to describe a cubic

- Positions of 4 points
  - u = 0, 1/3, 2/3, 1
  - Easy for 1 segment, hard to make connections
- Position & derivative at beginning and end
  - Hermite form



More ways to describe cubics

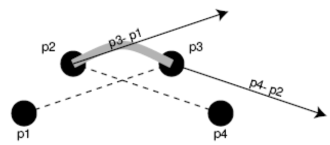
- Natural Cubics
  - "smoothest" curve
  - C(2)
- Each piece:
  - u=0: position, 1st derivative, 2nd derivative
  - u=1: position
- Piece 2 looks at values of previous piece (at end)
  - Propagation
- Non-local control (change at beginning changes everything)

4 Desirable Properties of Curves

- Interpolating
- C(2)
- Local-Control
- Cubic
- You can have any 3, but not all 4
  - Not C(2) = Hermite, Catmull-Rom
  - Not Local Control = Natural Cubics
  - Not Cubic = higher degree hermite
  - Not Interpolating = wait until next lectures (bezier, b-splines)

Cardinal Cubics
Catmull-Rom Splines

- Interpolate points
- Each segment interpolates p_1 and p_2
  - u=0, p_1 – derivative is k (p_2 – p_0)
  - u=1, p_2 – derivative is k (p_3 – p_1)
  - K = ½ for Catmull-Rom



Cardinal Interpolation

- $k = \frac{1}{2}(1-t)$ t = tension (0 for Catmull-Rom)