Lecture 4 – Sampling Theory

CS559 2007
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Notes not for reference, not for display

Outline

Last time
- Intensity/Quantization
- Gamma Correction
- Discrete sensing and displays
- Point samples
- Sampling, Reconstruction, Resampling, Aliasing

This time
- What can go wrong with sampling
- Sampling Theory Intuitions
- Some sampling theory math
- Sampling theory without the math

Bad sampling is bad

- Miss small things between samples

Get really weird results

- Sample a checkerboard
  - Look at a sampled picture
  - Too few samples
    - Get all black
    - Get all white
    - Get weird patterns
      - Aliasing
      - Moiré
    - Arbitrary algorithm decision gives very different answers!
- Imagine resampling

Demonstration ratios: 4/6 (here) = 2/3

Ugly

- Imagine line drawing
- Jaggies
- Crawlies
  - Small change causes jump
  - Smooth motion becomes jumpy

Dealing with discretization

- Sampling
  - Understand what information we are throwing away
- Reconstruction
  - Recreate as well as possible from the samples
- Re-Sampling
  - Transform the image
- Signal Processing / Image Processing
- Consider the 1D case first since its easier
Intuition

• Too few samples = BAD
• Sampling rate depends on the thing you’re sampling
• Need to sample close enough to get smallest object
• Need to limit small objects to be big enough that they aren’t missed

A different intuition

• Not really point sampling
  – Measurements average over a finite range
  – Displays make finite dots
• Need to model these
  – Sampling filters, reconstruction filters
  – Averages over regions -> Convolution (generalized)
• Need to be realistic about what they mean
  – Can’t see everything (too small, …)
• Sampling theory gives a nice mathematics for this!

Point sampling in 1D

• Only record samples
• Don’t know what happens in between samples
• Given the samples, don’t know what really happened!

Reconstruction from Sampling

• Can’t localize events
  – Bigger problems than that
• No idea! Signal could be anything
• Without additional information, we’re guessing as to what the signal is
• But what additional info?

Sampling Intuitions

• Reconstruct the “smoothest” signal that makes sense from samples
• If signal is “smooth enough”, sampling will give something we can reconstruct
• If signal is not “smooth”, sampling will give something that will reconstruct to something else
  – Aliasing
• But how do we define “smooth”?

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Signal processing

- Need better “language” for talking about signals
- Idea: represent signals in a different way
- Up till now: time domain (graph against time)
  - Good for asking “what does signal do at time X”
- New idea: frequency domain
  - Good for talking about how smooth signals are
- Different view of the same thing

Frequency Domain

- Fourier Theorem:
  - Any periodic signal can be represented as a sum of sine and cosine waves with harmonic frequencies
  - If one function has frequency $f$, then its harmonics are function with frequency $nf$ for integer $n$
  - Extensions to non-periodic signals later
  - Also works in any dimension (e.g. 2 for images, 3, …)
- Example: box

Example: Box (Square Wave)

- 1 cosine – bad
- More cosines, better approx

$$f(x) = \begin{cases} 
1 & |x| \leq \frac{1}{2} \\
0 & |x| > \frac{1}{2} 
\end{cases}$$

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi nx)}{\pi nx} \cos(2\pi nx)$$

Intuitions

- Low frequencies are smooth
  - High frequencies change fast, are not smooth
- If a signal can be made of only low frequencies, it is smooth
- If a signal has sharp changes, it will require high frequencies to represent
Fourier Transform

- $F(\omega)$ is the Fourier Transform of $f(t)$
  - A different representation of the same signal
  - Express as sums of sines and cosines
- To get $f(t)$ back you use the Inverse Fourier Transform
- You don’t need to know how to compute them

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx$$

Qualitative Properties

- The spectrum of a function tells us the relative amounts of high and low frequencies
  - Sharp edges give high frequencies
  - Smooth variations give low frequencies
- A function is **bandlimited** if its spectrum has no frequencies above a maximum limit
  - sin, cos are band limited
  - Box, Gaussian, etc are not
- To band-limit a signal we **low-pass filter** it

Sampling Theorem (intuition)

- High frequencies get lost
  - Can only sample band limited signals
- Sampling rate must be 2 times higher than signal
- Signal must be half frequency of sample rate
  - Otherwise, signal can “turn around” between samples
- Nyquist rate
  - 2x highest frequency in signal

Sampling Theorem

- If your signal is bandlimited
- And you know what the band limit is
- And you sample at (at least) twice that frequency
  - Above the Nyquist rate
- Then – you can reconstruct your signal **EXACTLY**!

- Caveat
  - Ideal reconstruction requires perfect band limiting in both sampling and reconstruction

Sampling theory in practice

- When you’re sampling- **PREFILTER**
  - Make sure no high frequencies
  - Need to remove them BEFORE sampling
  - Otherwise, aliasing
  - Filtering effectively means blurring
- When you’re reconstructing – **FILTER**
  - View as a spike chain (remove HF)
  - Filtering effectively means interpolating

Theory vs. Practice

<table>
<thead>
<tr>
<th>Theory</th>
<th>Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Properly sampled original</td>
<td>Who knows about source?</td>
</tr>
<tr>
<td>Know bandlimit</td>
<td>Assume that its OK?</td>
</tr>
<tr>
<td>Band-limit signals</td>
<td>Ideal LPF not practical</td>
</tr>
<tr>
<td>Use Ideal Filters</td>
<td>Use approximations</td>
</tr>
<tr>
<td>Ideal Reconstructions</td>
<td>Tradeoffs for “ideal”</td>
</tr>
</tbody>
</table>
  - Might look blurry
  - Might want aliasing (sharpness)
  - Care about efficiency
What is a filter anyway?

- Frequency filters
  - Add remove different frequencies
- Multiplication in frequency means CONVOLUTION in time/space
- Continuous and Discrete Convolutions