Question 1

1. \( E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \)

2. \( E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

3. \( E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \)

Question 2

There are several ways to solve this, including Gauss Elimination or computing the inverse of the matrix. The solution is

\[ x_1 = 1 \]
\[ x_2 = 2 \]
\[ x_3 = 3 \]
\[ x_4 = 4 \]

Question 3

1. Determinant \( x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_2 y_1 - x_3 y_2 \)

2. If the points are collinear, then the determinant is zero.

Suppose that the points are co-linear. Then either the line is vertical (so all the x's have the same value, call it c), or the line is \( y = mx + b \), and we can write the equation for three points as \( y_1 = m \cdot x_1 + b \), \( y_2 = m \cdot x_2 + b \), and \( y_3 = m \cdot x_3 + b \).

We can plug both of these cases into the matrix determinant formula and see that they are equal to zero.

\[ \text{Det} \begin{bmatrix} c & c & c \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} = cy_2 - cy_1 + cy_3 - cy_2 + cy_1 - cy_3 = 0 \]

and

\[ \text{Det} \begin{bmatrix} x_1 \text{ \( (mx_1 + b) \)} & x_2 \text{ \( (mx_2 + b) \)} & x_3 \text{ \( (mx_3 + b) \)} \\ 1 & 1 & 1 \end{bmatrix} \]

\[ = x_1(mx_2 + b) - x_2(mx_1 + b) + x_2(mx_3 + b) - x_3(mx_2 + b) + x_3(mx_1 + b) - x_1(mx_3 + b) \]

after a little term rearrangement \( b(x_1 + x_2 + x_3 - x_1 - x_2 - x_3) + m(x_1 x_2 + x_2 x_3 + x_3 x_1 - x_1 x_3 - x_2 x_1 - x_3 x_2) = 0 \)
3. If the determinant is zero, then the points are collinear

All of the line segments are connected, so to show that the points are co-linear, we need to show that the slopes are the same.

Plugging \(x_1, y_1, x_2, y_2, x_3, y_3\) into the determinant formula, and setting it equal to zero, we get

\[ x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0 \]

Now, a little re-arranging gives

\[ x_1y_2 - x_1y_3 + x_2y_3 = y_1x_2 - y_1x_3 + y_2x_3 \]

Then the clever step, subtract \(x_2y_2\) from both sides, which gives

\[ x_1y_2 - x_1y_3 + x_2y_3 - x_2y_2 = y_1x_2 - y_1x_3 + y_2x_3 - x_2y_2 \]

Which can be factored into \((x_1 - x_2)(y_2 - y_3) = (y_1 - y_2)(x_2 - x_3)\) **

which can be re-arranged into:

\[ (x_1 - x_2)/(y_1 - y_2) = (x_2 - x_3)/(y_2 - y_3) \]

showing that the slopes are indeed the same for these two segments. Except when the lines are horizontal (or vertical if you didn’t put the slopes upside down as I did in the last equation), so we can’t divide by \((y_1 - y_2)\) or \((y_2 - y_3)\) (since they’d be zero). So supposed one of them (say \(y_1 - y_2\)) is zero, Plug that into **, to get

\[ (x_1 - x_2)(y_2 - y_3) = 0, \]

which means either \(x_1 - x_2\) is zero or \(y_2 - y_3\) is zero. In the former case, 2 points are in the same place, so clearly the 3 points are collinear. In the latter case, both lines are horizontal, so their slopes are the same.

**Question 4**

1. **M matrix and one vector**

\(Ax\) needs \(k\) multiplications, \(k - 1\) additions for each element in the column. Totally, we will need \(k^2\) multiplications, \(k(k - 1)\) additions

\(A_1A_2\) need the same \(k\) multiplications, \(k - 1\) addition for each element in the array. Totally, we will need \(k^3\) multiplications, \(k^2(k - 1)\) additions

Thus the left to right will cost us \((n - 1)\) array multiplication and one array vector multiplication. It will be

\( (n - 1) * k^3 + k^2 \) multiplications and \( (n - 1) * k^2(k - 1) + k(k - 1) \) additions

From right to left will cost us \(n\) vector multiplication. It will be \(n * k^2\) multiplications and \(n * k(k - 1)\) additions

That’s why right to left is faster

How many times faster \(\frac{(n-1)k^3+k^2}{nk^2} = \frac{k(n-1)+1}{n}\)

2. For simplification, the following we only consider the multiplication only

If we do right to left, we need to repeat the process \(m\) times the cost will be \(m * n * k^2\)

If we do it left to right and store the result matrix, we need to do matrix multiplication, \((n - 1) * k^3\), and \(m\) matrix vector multiplication, \(m * k^2\). Totally, it will be \((n - 1) * k^3 + m * k^2\)

Faster \(\frac{(n-1)k^3+m*k^2}{nk^2} = \frac{k(n-1)+m}{n}\)

If \(\frac{k(n-1)+m}{n} > 1\), the right to left is faster. If \(\frac{k(n-1)+m}{n} < 1\), the left to right is faster

**Question 5**  

The vector normal to this plane is \((2, 2, 1)\). The shortest path from a point to the plane is along the normal direction (consider the triangle inequality), so the closest point to the origin is the intersection of a line passing through the origin in this direction and the plane. So we define the line as \((2t, 2t, t)\), plug this into the plane equation to get:

\[ \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \]

Since the distance from the origin to this point is \(\sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2} < 1\), this point is inside the unit sphere. Therefore, the plane must intersect the unit sphere.
Question 6

• Answers
Since the matrix is orthonormal, the dot product between any column and between any row must be zero.

Dot products between rows.
(1, 0, 0) · (r, s, s) = 0 ⇒ r = 0
(1, 0, 0) · (x, y, z) = 0 ⇒ x = 0
(0, s, s) · (0, y, z) = 0 ⇒ s(y + z) = 0

Dot products between columns
(0, s, y) · (0, s, z) = 0 ⇒ s^2 + yz = 0

Unit length in all columns and row.
2s^2 = 1 ⇒ s = ±\frac{1}{\sqrt{2}}

y^2 + z^2 = 1

From s(y + z) = 0, s^2 + yz = 0, and s ≠ 0, we get that y = −z

With y^2 + z^2 = 1 and y = −z, we get either y = \frac{1}{\sqrt{2}}, z = −\frac{1}{\sqrt{2}} or y = −\frac{1}{\sqrt{2}}, z = \frac{1}{\sqrt{2}}

We can four different answers
r = 0, s = \frac{1}{\sqrt{2}}, x = 0, y = \frac{1}{\sqrt{2}}, z = −\frac{1}{\sqrt{2}}
r = 0, s = \frac{1}{\sqrt{2}}, x = 0, y = −\frac{1}{\sqrt{2}}, z = \frac{1}{\sqrt{2}}
r = 0, s = −\frac{1}{\sqrt{2}}, x = 0, y = \frac{1}{\sqrt{2}}, z = −\frac{1}{\sqrt{2}}
r = 0, s = −\frac{1}{\sqrt{2}}, x = 0, y = −\frac{1}{\sqrt{2}}, z = \frac{1}{\sqrt{2}}

• Right handed
The cross product of the first and second row will be the third.

r = 0, s = \frac{1}{\sqrt{2}}, x = 0, y = −\frac{1}{\sqrt{2}}, z = \frac{1}{\sqrt{2}}
r = 0, s = −\frac{1}{\sqrt{2}}, x = 0, y = \frac{1}{\sqrt{2}}, z = −\frac{1}{\sqrt{2}}

• Left handed
The cross product of the first and second row will be the negative third.

r = 0, s = \frac{1}{\sqrt{2}}, x = 0, y = \frac{1}{\sqrt{2}}, z = −\frac{1}{\sqrt{2}}
r = 0, s = −\frac{1}{\sqrt{2}}, x = 0, y = −\frac{1}{\sqrt{2}}, z = \frac{1}{\sqrt{2}}

Question 7

1. Geometric intuition to \textbf{n} \textbf{n} will be perpendicular to the plane passing through \textbf{p}_(0), \textbf{p}_(1), \textbf{p}_(2)

2. What value is \textbf{n}(\textbf{p}_(3) − \textbf{p}_(0)) From previous question, \textbf{n} is perpendicular to all vectors which is parallel to that plane and thus their dot product will be zero ⇒ \textbf{n} · \textbf{v} where \textbf{v} is parallel to the plane.

Since \textbf{p}_(3) − \textbf{p}_(0) is a vector parallel to the plane, We can get \textbf{n}(\textbf{p}_(3) − \textbf{p}_(0)) = 0

3. Plane equation
Since \textbf{p} is on the plane and \textbf{n}(\textbf{p} − \textbf{p}_(0)) = 0, we can get

n_x(x − p_x) + n_y(y − p_y) + n_z(z − p_z) = 0
n_x x + n_y y + n_z z − (n_x p_x + n_y p_y + n_z p_z) = 0

Thus a = n_x, b = n_y, c = n_z, d = −(n_x p_x + n_y p_y + n_z p_z)